PRICING OF INFLATION-INDEXED DERIVATIVES

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Stylized facts

- Inflation-indexed bonds have been issued since the 80's, but it is only in the very last years that these bonds, and inflation-indexed derivatives in general, have become quite popular.
- Inflation is defined as the percentage increment of a reference index, the Consumer Price Index (CPI), which is a basket of good and services.
- In theory, but also in practice, inflation can become negative.
- Banks typically issues inflation-linked bonds, where a zero-strike floor is offered in conjunction with the "pure" bond (to grant positive coupons, the inflation rate is floored at zero).
- Floors with low strikes are the most actively traded options on inflation rates. Other extremely popular derivatives are inflation-indexed swaps.

Stylized facts (cont'd)

- Inflation-indexed derivatives require a specific model to be valued.
- Main references: Barone and Castagna (1997), van Bezooyen et al. (1997), Hughston (1998), Cairns (2000) and Jarrow and Yildirim (2003).
- Inflation derivatives are priced with a foreign-currency analogy (the pricing is equivalent to that of a cross-currency interest-rate derivative).
- What is typically modelled is the evolution of the instantaneous nominal and real rates and of the CPI (interpreted as the "exchange rate" between the nominal and real economies).
- The real rate one models is the rate we can lock in by suitably trading in inflation swaps. The true real rate will be only known at the end of the corresponding period (as soon as the CPI's value is known).

Purpose and outline of the talk

- Our purpose is to price analytically, and consistently with no arbitrage, inflation-indexed swaps and options.

- We start by introducing the two main types of inflation swaps.

- We first apply the JY model and then propose two different market models. We derive closed-form formulas in all the presented cases.

- We then introduce inflation caps and floors, and we price them both under the JY model and under our second market model.

- The advantage of our market-model approach is in terms of a better understanding of the model parameters and of a more accurate calibration to market data.

Some notations and definitions

We denote by $I(t)$ the CPI's value at time t.

We use the subscripts n and r to denote quantities in the nominal and real economies, respectively.

The zero-coupon bond prices at time t for maturity T in the nominal and real economies are denoted, respectively, by $P_n(t,T)$ and $P_r(t,T)$.

The instantaneous forward rates at time t for maturity T are defined by

$$f_x(t,T) = -\frac{\partial \ln P_x(t,T)}{\partial T}, \quad x \in \{n,r\}$$

and the corresponding instantaneous short rates by

$$n(t) = f_n(t,t),$$
$$r(t) = f_r(t,t).$$

Some notations and definitions (cont'd)

Given the future time interval $[T_{i-1}, T_i]$, the related forward LIBOR rates, at time t, are

$$F_x(t; T_{i-1}, T_i) = \frac{P_x(t, T_{i-1}) - P_x(t, T_i)}{\tau_i P_x(t, T_i)}, \quad x \in \{n, r\},$$

where τ_i is the year fraction for $[T_{i-1}, T_i]$.

We denote by Q_n and Q_r the nominal and real risk-neutral measures, respectively, and by E_x the expectation associated to Q_x, $x \in \{n, r\}$.

We denote by Q_n^T the nominal T-forward measure and by E_n^T the associated expectation.

Finally, the σ-algebra generated by the relevant processes up to time t is denoted by $\mathcal{F}_t$.

Historical plots of CPIs

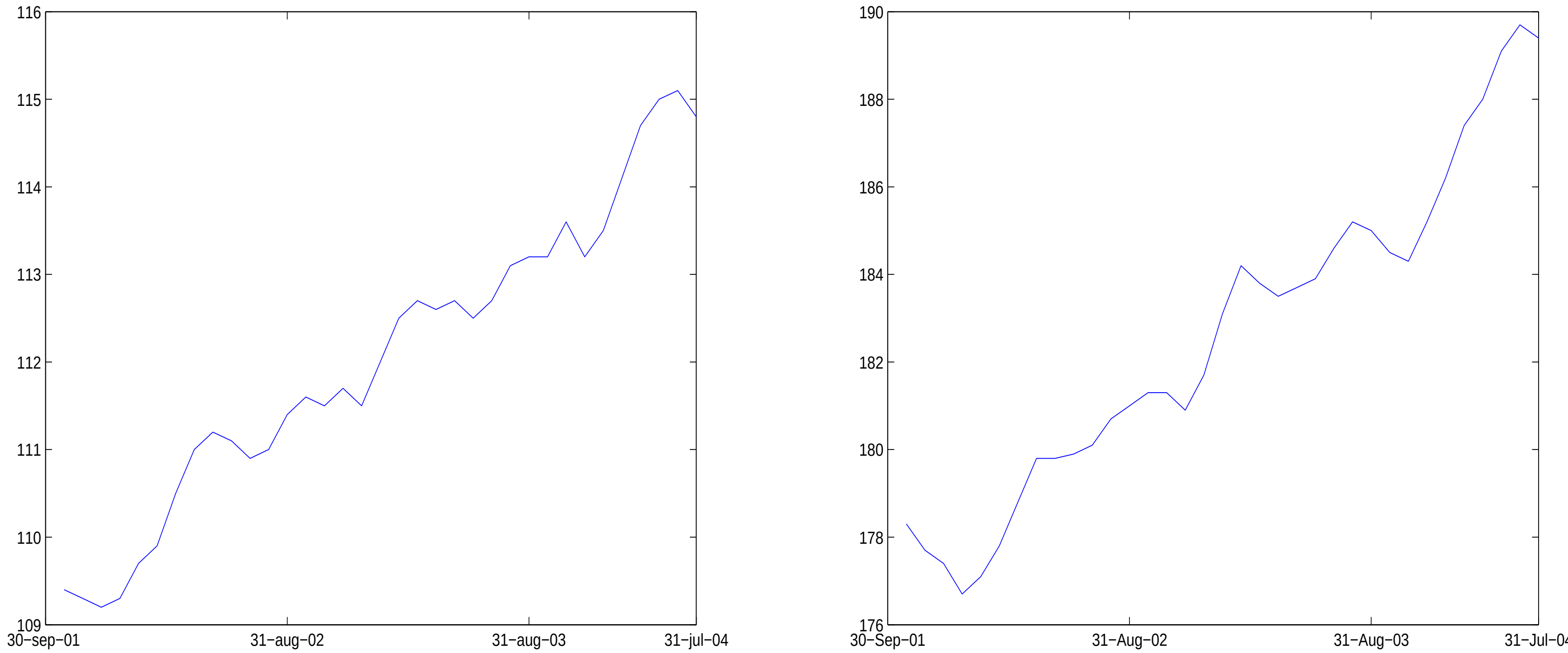


Figure 1: Left: EUR CPI Unrevised Ex-Tobacco. Right: USD CPI Urban Consumers NSA. Monthly closing values from 30-Sep-01 to 21-Jul-04.

Inflation-indexed swaps

Given a set of dates $T_1, \ldots, T_M$, an Inflation-Indexed Swap (IIS) is a swap where, on each payment date, Party A pays Party B the inflation rate over a predefined period, while Party B pays Party A a fixed rate.

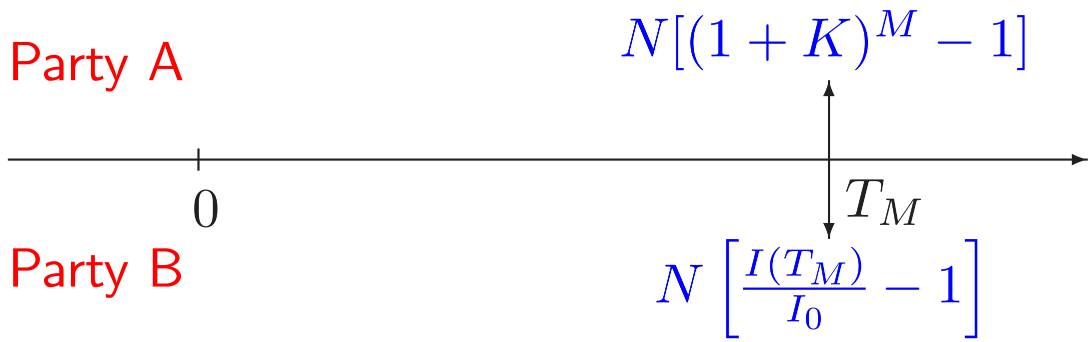
The inflation rate is calculated as the percentage return of the CPI index I over the time interval $[t, T]$ it applies to:

$$i(t,T) := \frac{I(T)}{I(t)} - 1.$$

Two are the main IIS traded in the market:

- the zero coupon (ZC) swap;
- the year-on-year (YY) swap.

Zero-coupon inflation-indexed swaps



In a ZCIIS, at time $T_M = M$ years, Party B pays Party A the fixed amount

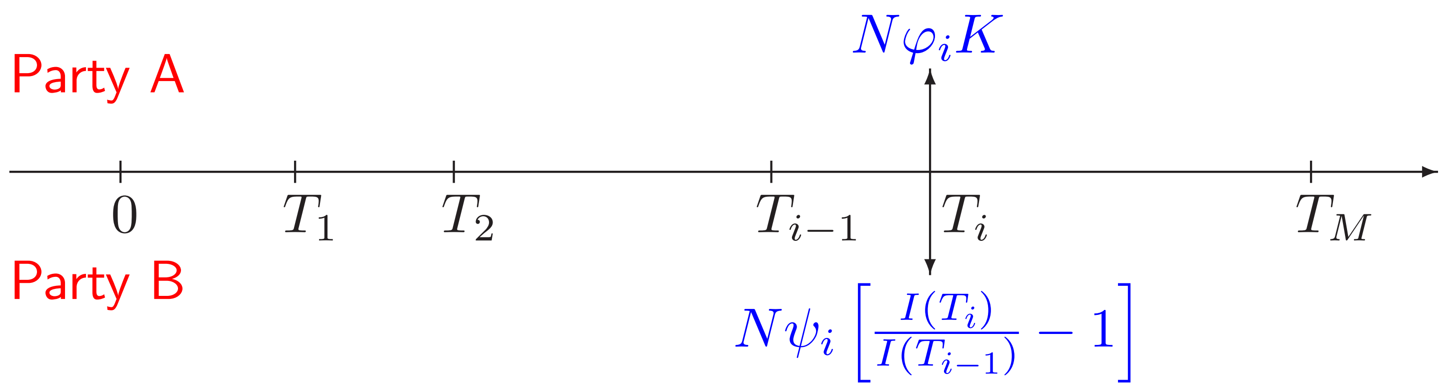
$$N[(1+K)^M - 1],$$

where K and N are, respectively, the contract fixed rate and the contract nominal value.

Party A pays Party B, at the final time T_M, the floating amount

$$N\left[\frac{I(T_M)}{I_0} - 1\right].$$

Year-on-year inflation-indexed swaps



In a YYIIS, at each time T_i, Party B pays Party A the fixed amount

$$N\varphi_i K,$$

while Party A pays Party B the (floating) amount

$$N\psi_i \left[\frac{I(T_i)}{I(T_{i-1})} - 1\right],$$

where φ_i and ψ_i are, respectively, the fixed- and floating-leg year fractions for the interval $[T_{i-1}, T_i]$, $T_0 := 0$ and N is again the swap nominal value.

Pricing of a ZCIIS

Standard no-arbitrage pricing theory implies that the value at time t, $0 \leq t < T_M$, of the inflation-indexed leg of the ZCIIS is

$$\mathbf{ZCIIS}(t, T_M, I_0, N) = N E_n \left\{ e^{-\int_t^{T_M} n(u)\,du} \left[\frac{I(T_M)}{I_0} - 1 \right] \Big| \mathcal{F}_t \right\}.$$

The foreign-currency analogy implies that, for each $t < T$:

$$I(t) P_r(t,T) = I(t) E_r \left\{ e^{-\int_t^T r(u)\,du} \Big| \mathcal{F}_t \right\} = E_n \left\{ e^{-\int_t^T n(u)\,du} I(T) \Big| \mathcal{F}_t \right\},$$

namely, the nominal price of a real zero-coupon bond equals the nominal price of the contract paying one unit of the CPI index at the bond maturity.

We thus have:

$$\mathbf{ZCIIS}(t, T_M, I_0, N) = N \left[\frac{I(t)}{I_0} P_r(t, T_M) - P_n(t, T_M) \right].$$

Pricing of a ZCIIS (cont'd)

The ZCIIS price is therefore model-independent: it is not based on specific assumptions on the interest rates evolution, but simply follows from the absence of arbitrage.

This result is extremely important since it enables us to strip, with no ambiguity, real zero-coupon bond prices from the quoted prices of zero-coupon inflation-indexed swaps.

The market quotes values of $K = K(T_M)$ for some given maturities T_M.

The ZCIIS corresponding to $(T_M, K(T_M))$ has zero value at time $t = 0$ if and only if

$$N[P_r(0,T_M) - P_n(0,T_M)] = NP_n(0,T_M)[(1+K(T_M))^M - 1]$$

$$\Rightarrow P_r(0,T_M) = P_n(0,T_M)(1+K(T_M))^M.$$

Pricing of a YYIIS

The valuation of a YYIIS is less straightforward and, as we shall see, requires the specification of an interest rate model.

The value at time $t < T_i$ of the YYIIS payoff at time T_i is

$$\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i E_n \left\{ e^{-\int_t^{T_i} n(u)\,du} \left[\frac{I(T_i)}{I(T_{i-1})} - 1 \right] \Big| \mathcal{F}_t \right\},$$

which, assuming $t < T_{i-1}$, can be calculated as

$$N\psi_i E_n \left\{ e^{-\int_t^{T_{i-1}} n(u)\,du} E_n \left[e^{-\int_{T_{i-1}}^{T_i} n(u)\,du} \left(\frac{I(T_i)}{I(T_{i-1})} - 1 \right) \Big| \mathcal{F}_{T_{i-1}} \right] \Big| \mathcal{F}_t \right\}.$$

The inner expectation is nothing but $\mathbf{ZCIIS}(T_{i-1}, T_i, I(T_{i-1}), 1)$

$$\Rightarrow N\psi_i E_n \left\{ e^{-\int_t^{T_{i-1}} n(u)\,du} [P_r(T_{i-1}, T_i) - P_n(T_{i-1}, T_i)] \Big| \mathcal{F}_t \right\}.$$

Pricing of a YYIIS (cont'd)

We thus have:

$$\textbf{YYIIS}(t) = N\psi_i E_n\left\{e^{-\int_t^{T_{i-1}} n(u)\,du} P_r(T_{i-1}, T_i)\Big|\mathcal{F}_t\right\} - N\psi_i P_n(t, T_i).$$

This last expectation can be viewed as the nominal price of a derivative paying off $P_r(T_{i-1}, T_i)$ at time T_{i-1}. If real rates were deterministic:

$$\begin{aligned} E_n\left\{e^{-\int_t^{T_{i-1}} n(u)\,du} P_r(T_{i-1}, T_i)\Big|\mathcal{F}_t\right\} &= P_r(T_{i-1}, T_i) P_n(t, T_{i-1}) \\ &= \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} P_n(t, T_{i-1}), \end{aligned}$$

which is the present value, in nominal terms, of the forward price of the real bond. In practice, however, real rates are stochastic and the above expected value is model dependent.

The JY model

Under the real-world probability space $(\Omega, \mathcal{F}, P)$, with associated filtration $\mathcal{F}_t$, Jarrow and Yildirim assumed that

$$df_n(t,T) = \alpha_n(t,T)\,dt + \varsigma_n(t,T)\,dW_n^P(t)$$
$$df_r(t,T) = \alpha_r(t,T)\,dt + \varsigma_r(t,T)\,dW_r^P(t)$$
$$dI(t) = I(t)\mu(t)\,dt + \sigma_I I(t)\,dW_I^P(t)$$

with $I(0) = I_0 > 0$, and $f_x(0,T) = f_x^{\mathrm{MKT}}(0,T)$, $x \in \{n, r\}$, where

- (W_n^P, W_r^P, W_I^P) is a Brownian motion with correlations $\rho_{n,r}$, $\rho_{n,I}$ and $\rho_{r,I}$;
- α_n, α_r and μ are adapted processes;
- ς_n and ς_r are deterministic functions;
- σ_I is a positive constant.

The JY model (cont'd)

Choosing the forward rate volatilities as

$$\varsigma_n(t,T) = \sigma_n\, e^{-a_n(T-t)}, \quad \varsigma_r(t,T) = \sigma_r\, e^{-a_r(T-t)},$$

where σ_n σ_r, a_n and a_r are positive constants, and using the equivalent formulation in terms of instantaneous short rates, we have the following.

Proposition. The Q_n-dynamics of n, r and I are, respectively,

$$dn(t) = [\vartheta_n(t) - a_n n(t)]\, dt + \sigma_n\, dW_n(t)$$

$$dr(t) = [\vartheta_r(t) - \rho_{r,I}\sigma_I\sigma_r - a_r r(t)]\, dt + \sigma_r\, dW_r(t)$$

$$dI(t) = I(t)[n(t) - r(t)]\, dt + \sigma_I I(t)\, dW_I(t)$$

where (W_n, W_r, W_I) is a Brownian motion with correlations $\rho_{n,r}$, $\rho_{n,I}$ and $\rho_{r,I}$, and

$$\vartheta_x(t) = \frac{\partial f_x(0,t)}{\partial T} + a_x f_x(0,t) + \frac{\sigma_x^2}{2a_x}(1 - e^{-2a_x t}), \quad x \in \{n, r\}.$$

Pricing of a YYIIS: the JY model

We remember that:

$$\textbf{YYIIS}(t) = N\psi_i E_n \left\{ e^{-\int_t^{T_{i-1}} n(u)\,du} P_r(T_{i-1}, T_i) \Big| \mathcal{F}_t \right\} - N\psi_i P_n(t, T_i)$$

$$= N\psi_i P_n(t, T_{i-1}) E_n^{T_{i-1}} \left\{ P_r(T_{i-1}, T_i) \Big| \mathcal{F}_t \right\} - N\psi_i P_n(t, T_i),$$

and also remember the zero-coupon bond price formula in the Hull and White (1994) model:

$$P_r(t, T) = A_r(t, T) e^{-B_r(t,T) r(t)},$$

$$B_r(t, T) = \frac{1}{a_r} \left[1 - e^{-a_r(T-t)} \right],$$

$$A_r(t, T) = \frac{P_r^M(0, T)}{P_r^M(0, t)} \exp\left\{ B_r(t, T) f_r^M(0, t) - \frac{\sigma^2}{4a_r}(1 - e^{-2a_r t}) B_r(t, T)^2 \right\}.$$

Pricing of a YYIIS: the JY model (cont'd)

Since the real instantaneous rate evolves under $Q_n^{T_{i-1}}$ according to

$$dr(t) = [-\rho_{n,r}\sigma_n\sigma_r B_n(t,T_{i-1}) + \vartheta_r(t) - \rho_{r,I}\sigma_I\sigma_r - a_r r(t)]\,dt + \sigma_r\,dW_r^{T_{i-1}}(t),$$

$r(T_{i-1})$ remains a normal random variable, and hence the real bond price $P_r(T_{i-1},T_i)$ is lognormally distributed also under $Q_n^{T_{i-1}}$.

After some tedious, but straightforward, algebra we finally obtain

$$\textbf{YYIIS}(t) = N\psi_i P_n(t,T_{i-1})\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{C(t,T_{i-1},T_i)} - N\psi_i P_n(t,T_i),$$

where

$$\begin{aligned} C(t,T_{i-1},T_i) =& \sigma_r B_r(T_{i-1},T_i)\Bigg[B_r(t,T_{i-1})\Big(\rho_{r,I}\sigma_I - \tfrac{1}{2}\sigma_r B_r(t,T_{i-1}) \\ &+ \frac{\rho_{n,r}\sigma_n}{a_n+a_r}(1+a_r B_n(t,T_{i-1}))\Big) - \frac{\rho_{n,r}\sigma_n}{a_n+a_r}B_n(t,T_{i-1})\Bigg]. \end{aligned}$$

Pricing of a YYIIS: the JY model (cont'd)

The value at time t of the inflation-indexed leg of the swap is simply obtained by summing up the values of all floating payments. We thus get

$$\mathbf{YYIIS}(t, \mathcal{T}, \Psi, N) = N\psi_{\iota(t)}\left[\frac{I(t)}{I(T_{\iota(t)-1})}P_r(t, T_{\iota(t)}) - P_n(t, T_{\iota(t)})\right]$$

$$+ N \sum_{i=\iota(t)+1}^{M} \psi_i \left[P_n(t, T_{i-1})\frac{P_r(t, T_i)}{P_r(t, T_{i-1})}\, e^{C(t,T_{i-1},T_i)} - P_n(t, T_i)\right],$$

where $\mathcal{T} := \{T_1, \ldots, T_M\}$, $\Psi := \{\psi_1, \ldots, \psi_M\}$ and $\iota(t) = \min\{i : T_i > t\}$. In particular at $t = 0$,

$$\mathbf{YYIIS}(0) = \sum_{i=1}^{M} \psi_i P_n(0, T_i) \left[\frac{1 + \tau_i F_n(0; T_{i-1}, T_i)}{1 + \tau_i F_r(0; T_{i-1}, T_i)}\, e^{C(0,T_{i-1},T_i)} - 1\right].$$

Pricing of a YYIIS: the LIBOR market model

For an alternative pricing of the above YYIIS, we notice that

$$P_n(t,T_{i-1})E_n^{T_{i-1}}\left\{P_r(T_{i-1},T_i)\big|\mathcal{F}_t\right\} = P_n(t,T_i)E_n^{T_i}\left\{\frac{P_r(T_{i-1},T_i)}{P_n(T_{i-1},T_i)}\Big|\mathcal{F}_t\right\}$$

$$= P_n(t,T_i)E_n^{T_i}\left\{\frac{1+\tau_i F_n(T_{i-1};T_{i-1},T_i)}{1+\tau_i F_r(T_{i-1};T_{i-1},T_i)}\Big|\mathcal{F}_t\right\}.$$

It seems natural, therefore, to resort to a (lognormal) LIBOR model, for both nominal and real rates.

Since $I(t)P_r(t,T_i)$ is the price of an asset in the nominal economy, the forward CPI

$$\mathcal{I}_i(t) := I(t)\frac{P_r(t,T_i)}{P_n(t,T_i)}$$

is a martingale under $Q_n^{T_i}$.

Pricing of a YYIIS: the LIBOR market model (cont'd)

We assume:

$$d\mathcal{I}_i(t) = \sigma_{I,i}\mathcal{I}_i(t)\, dW_i^I(t).$$

where $\sigma_{I,i}$ is a positive constant and W_i^I is a $Q_n^{T_i}$-Brownian motion.

Assuming also that both nominal and real forward rates follow a lognormal LIBOR market model, the foreign-currency analogy implies that, under $Q_n^{T_i}$,

$$dF_n(t; T_{i-1}, T_i) = \sigma_{n,i} F_n(t; T_{i-1}, T_i)\, dW_i^n(t),$$

$$dF_r(t; T_{i-1}, T_i) = F_r(t; T_{i-1}, T_i)\big[-\rho_{I,r,i}\sigma_{I,i}\sigma_{r,i}\, dt + \sigma_{r,i}\, dW_i^r(t)\big],$$

where $\sigma_{n,i}$ and $\sigma_{r,i}$ are positive constants, W_i^n and W_i^r are two standard Brownian motions with instantaneous correlation ρ_i, and $\rho_{I,r,i}$ is the instantaneous correlation between $\mathcal{I}_i(\cdot)$ and $F_r(\cdot; T_{i-1}, T_i)$.

Pricing of a YYIIS: the LIBOR market model (cont'd)

The last expectation can now be easily calculated with a numerical integration by noting that, under $Q_n^{T_i}$ and conditional on $\mathcal{F}_t$, the pair

$$(X_i, Y_i) = \left(\ln \frac{F_n(T_{i-1}; T_{i-1}, T_i)}{F_n(t; T_{i-1}, T_i)}, \ln \frac{F_r(T_{i-1}; T_{i-1}, T_i)}{F_r(t; T_{i-1}, T_i)} \right)$$

is distributed as a bivariate normal random variable with mean vector and variance-covariance matrix respectively given by

$$M_{X_i,Y_i} = \begin{bmatrix} \mu_{x,i}(t) \\ \mu_{y,i}(t) \end{bmatrix}, \quad V_{X_i,Y_i} = \begin{bmatrix} \sigma_{x,i}^2(t) & \rho_i \sigma_{x,i}(t) \sigma_{y,i}(t) \\ \rho_i \sigma_{x,i}(t) \sigma_{y,i}(t) & \sigma_{y,i}^2(t) \end{bmatrix},$$

where

$$\mu_{x,i}(t) = -\tfrac{1}{2}\sigma_{n,i}^2 (T_{i-1} - t), \quad \sigma_{x,i}(t) = \sigma_{n,i}\sqrt{T_{i-1} - t},$$

$$\mu_{y,i}(t) = \left[-\tfrac{1}{2}\sigma_{r,i}^2 - \rho_{I,r,i}\sigma_{I,i}\sigma_{r,i} \right] (T_{i-1} - t), \quad \sigma_{y,i}(t) = \sigma_{r,i}\sqrt{T_{i-1} - t}.$$

Pricing of a YYIIS: the LIBOR market model (cont'd)

It is well known that the joint density $f_{X_i,Y_i}(x,y)$ can be decomposed as

$$f_{X_i,Y_i}(x,y) = f_{X_i|Y_i}(x,y) f_{Y_i}(y),$$

where

$$f_{X_i|Y_i}(x,y) = \frac{1}{\sigma_{x,i}(t)\sqrt{2\pi}\sqrt{1-\rho_i^2}} \exp\left[-\frac{\left(\frac{x-\mu_{x,i}(t)}{\sigma_{x,i}(t)} - \rho_i \frac{y-\mu_{y,i}(t)}{\sigma_{y,i}(t)}\right)^2}{2(1-\rho_i^2)} \right]$$

$$f_{Y_i}(y) = \frac{1}{\sigma_{y,i}(t)\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y-\mu_{y,i}(t)}{\sigma_{y,i}(t)}\right)^2 \right].$$

The expectation can thus be calculated as (we set $F_x(t) := F_x(t; T_{i-1}, T_i)$)

$$\int_{-\infty}^{+\infty} \frac{1}{1+\tau_i F_r(t)\, e^y} \left[\int_{-\infty}^{+\infty} (1+\tau_i F_n(t) e^x)\, f_{X_i|Y_i}(x,y)\, dx \right] f_{Y_i}(y)\, dy$$

Pricing of a YYIIS: the LIBOR market model (cont'd)

The value at time t of the inflation-indexed leg of the swap is thus given by

$$\textbf{YYIIS}(t,\mathcal{T},\Psi,N) = N\psi_{\iota(t)}\left[\frac{I(t)}{I(T_{\iota(t)-1})}P_r(t,T_{\iota(t)}) - P_n(t,T_{\iota(t)})\right]$$

$$+ N\sum_{i=\iota(t)+1}^{M}\psi_i P_n(t,T_i)\left[\int_{-\infty}^{+\infty}\frac{1+\tau_i F_n(t)\,e^{\rho_i\sigma_{x,i}(t)z-\frac{1}{2}\sigma_{x,i}^2(t)\rho_i^2}}{1+\tau_i F_r(t)\,e^{\mu_{y,i}(t)+\sigma_{y,i}(t)z}}\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\,dz - 1\right].$$

N.B. In theory, the volatilities $\sigma_{I,i}$ can not be constant for each i, see Schlögl (2002). In practice, however, they are approximately constant.

In particular at $t=0$, $\textbf{YYIIS}(0,\mathcal{T},\Psi,N) =$

$$= N\sum_{i=1}^{M}\psi_i P_n(0,T_i)\left[\int_{-\infty}^{+\infty}\frac{1+\tau_i F_n(0)\,e^{\rho_i\sigma_{x,i}(0)z-\frac{1}{2}\sigma_{x,i}^2(0)\rho_i^2}}{1+\tau_i F_r(0)\,e^{\mu_{y,i}(0)+\sigma_{y,i}(0)z}}\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\,dz - 1\right]$$

Pricing of a YYIIS: the LIBOR market model (cont'd)

This YYIIS price depends on: the (instantaneous) volatilities of nominal and real forward rates and their correlations, for $i = 2, \ldots, M$; the correlations between real forward rates and forward inflation indices, again for $i = 2, \ldots, M$.

Compared with the JY expression, this last formula looks more complicated both in terms of input parameters and in terms of the calculations involved.

However, one-dimensional numerical integrations are not so cumbersome. Moreover, as is typical in a market model, the input parameters can be determined more easily than in the previous short-rate approach.

Both approaches seen so far have the drawback that the volatility of real rates may be hard to estimate. This is why we propose a second market-model approach, which enables us to overcome this estimation issue.

Pricing of a YYIIS: a second market model

Applying the definition of forward CPI and using the fact that $\mathcal{I}_i$ is a martingale under $Q_n^{T_i}$, we can also write, for $t < T_{i-1}$,

$$\begin{aligned}\textbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) &= N\psi_i P(t, T_i) E_n^{T_i}\left\{\frac{I(T_i)}{I(T_{i-1})} - 1\Big|\mathcal{F}_t\right\}\\ &= N\psi_i P(t, T_i) E_n^{T_i}\left\{\frac{\mathcal{I}_i(T_i)}{\mathcal{I}_{i-1}(T_{i-1})} - 1\Big|\mathcal{F}_t\right\}\\ &= N\psi_i P(t, T_i) E_n^{T_i}\left\{\frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} - 1\Big|\mathcal{F}_t\right\}.\end{aligned}$$

We recall that, under $Q_n^{T_i}$,

$$d\mathcal{I}_i(t) = \sigma_{I,i}\mathcal{I}_i(t)\, dW_i^I(t)$$

and that an analogous evolution holds for $\mathcal{I}_{i-1}$ under $Q_n^{T_{i-1}}$.

Pricing of a YYIIS: a second market model (cont'd)

The dynamics of $\mathcal{I}_{i-1}$ under $Q_n^{T_i}$ are

$$d\mathcal{I}_{i-1}(t) = -\mathcal{I}_{i-1}(t)\sigma_{I,i-1}\frac{\tau_i\sigma_{n,i}F_n(t;T_{i-1},T_i)}{1+\tau_i F_n(t;T_{i-1},T_i)}\rho_{I,n,i}\,dt + \sigma_{I,i-1}\mathcal{I}_{i-1}(t)\,dW_{i-1}^I(t),$$

where $\sigma_{I,i-1}$ is a positive constant, W_{i-1}^I is a $Q_n^{T_i}$-Brownian motion with $dW_{i-1}^I(t)\,dW_i^I(t) = \rho_{I,i}\,dt$, and $\rho_{I,n,i}$ is the instantaneous correlation between $\mathcal{I}_{i-1}(\cdot)$ and $F_n(\cdot;T_{i-1},T_i)$.

The evolution of $\mathcal{I}_{i-1}$, under $Q_n^{T_i}$, depends on the nominal rate $F_n(\cdot;T_{i-1},T_i)$.

To avoid unpleasant calculations, we freeze the above drift at its current time-t value, so that $\mathcal{I}_{i-1}(T_{i-1})$ conditional on $\mathcal{F}_t$ is lognormally distributed also under $Q_n^{T_i}$.

Pricing of a YYIIS: a second market model (cont'd)

Also the ratio $\mathcal{I}_i(T_{i-1})/\mathcal{I}_{i-1}(T_{i-1})$ conditional on $\mathcal{F}_t$ is lognormally distributed under $Q_n^{T_i}$. This leads to

$$E_n^{T_i}\left\{\frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})}\Big|\mathcal{F}_t\right\} = \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}\,e^{D_i(t)},$$

where

$$D_i(t) = \sigma_{I,i-1}\left[\frac{\tau_i\sigma_{n,i}F_n(t;T_{i-1},T_i)}{1+\tau_iF_n(t;T_{i-1},T_i)}\rho_{I,n,i} - \rho_{I,i}\sigma_{I,i} + \sigma_{I,i-1}\right](T_{i-1}-t),$$

so that

$$\textbf{YYIIS}(t,T_{i-1},T_i,\psi_i,N) = N\psi_iP_n(t,T_i)\left[\frac{P_n(t,T_{i-1})P_r(t,T_i)}{P_n(t,T_i)P_r(t,T_{i-1})}\,e^{D_i(t)} - 1\right].$$

N.B. This pricing method is equivalent to, but independently derived from, that of Belgrade, Benhamou and Koehler (2004).

Pricing of a YYIIS: a second market model (cont'd)

Finally, the value at time t of the inflation-indexed leg of the swap is

$$\mathbf{YYIIS}(t,\mathcal{T},\Psi,N) = N\psi_{\iota(t)}\left[\frac{I(t)}{I(T_{\iota(t)-1})}P_r(t,T_{\iota(t)}) - P_n(t,T_{\iota(t)})\right]$$
$$+ N\sum_{i=\iota(t)+1}^{M}\psi_i\left[P_n(t,T_{i-1})\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{D_i(t)} - P_n(t,T_i)\right].$$

In particular at $t=0$,

$$\mathbf{YYIIS}(0,\mathcal{T},\Psi,N) = N\sum_{i=1}^{M}\psi_i P_n(0,T_i)\left[\frac{1+\tau_i F_n(0;T_{i-1},T_i)}{1+\tau_i F_r(0;T_{i-1},T_i)}e^{D_i(0)} - 1\right].$$

Pricing of a YYIIS: a second market model (cont'd)

This YYIIS price depends on: the (instantaneous) volatilities of forward inflation indices and their correlations; the (instantaneous) volatilities of nominal forward rates; the instantaneous correlations between forward inflation indices and nominal forward rates.

This pricing formula looks pretty similar to that in the JY case and may be preferred to the one in the LIBOR market model, since it combines the advantage of a fully-analytical formula with that of a market-model approach.

Moreover, the correction term D does not depend on the volatility of real rates, which is typically difficult to estimate.

The only drawback is that the approximation it is based on may be rough for long maturities T_i. The formula, however, is exact when the correlations $\rho_{I,n,i}$ are set to zero and the terms D_i are simplified accordingly.

Inflation-indexed caplets

An Inflation-Indexed Caplet (IIC) is a call option on the inflation rate implied by the CPI index.

Analogously, an Inflation-Indexed Floorlet (IIF) is a put option on the same inflation rate.

In formulas, at time T_i, the IICF payoff is

$$N\psi_i \left[\omega\left(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa\right)\right]^+,$$

where κ is the IICF strike, ψ_i is the contract year fraction for the interval $[T_{i-1}, T_i]$, N is the contract nominal value, and $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet.

We set $K := 1 + \kappa$.

Inflation-indexed caplets (cont'd)

Standard no-arbitrage pricing theory implies that the value at time $t \leq T_{i-1}$ of the IICF at time T_i is

$$
\begin{aligned}
&\mathbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) \\
&= N\psi_i E_n \left\{ e^{-\int_t^{T_i} n(u)\,du} \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \Big| \mathcal{F}_t \right\} \\
&= N\psi_i P_n(t, T_i) E_n^{T_i} \left\{ \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \Big| \mathcal{F}_t \right\}.
\end{aligned}
$$

The pricing of a IICF is thus similar to that of a forward-start (cliquet) option.

We now derive analytical formulas both under the JY model and under our second market model approach.

Inflation-indexed caplets: the JY model

The assumption of Gaussian nominal and real rates leads to a CPI that is lognomally distributed under Q_n.

When we move to a (nominal) forward measure the type of distribution is preserved.

Hence, $\frac{I(T_i)}{I(T_{i-1})}$ conditional on $\mathcal{F}_t$ is lognormally distributed also under $Q_n^{T_i}$, and the IICF price can be calculated as follows.

If X is a lognormal random variable with $E(X) = m$ and Std$[\ln(X)] = v$, then

$$E\left\{[\omega(X-K)]^+\right\} = \omega m \Phi\left(\omega \frac{\ln \frac{m}{K} + \frac{1}{2}v^2}{v}\right) - \omega K \Phi\left(\omega \frac{\ln \frac{m}{K} - \frac{1}{2}v^2}{v}\right),$$

where Φ denotes the standard normal distribution function.

Inflation-indexed caplets: the JY model (cont'd)

The conditional expectation of $I(T_i)/I(T_{i-1})$ is immediately obtained through the price of a YYIIS:

$$E_n^{T_i}\left\{\frac{I(T_i)}{I(T_{i-1})}\Big|\mathcal{F}_t\right\} = \frac{P_n(t,T_{i-1})}{P_n(t,T_i)}\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}\,e^{C(t,T_{i-1},T_i)}.$$

The variance of the log of the ratio can be equivalently calculated under the (nominal) risk-neutral measure. We get:

$$\mathrm{Var}_n^{T_i}\left\{\ln\frac{I(T_i)}{I(T_{i-1})}\Big|\mathcal{F}_t\right\} = V^2(t,T_{i-1},T_i),$$

where, setting $\Delta T_i := T_i - T_{i-1}$,

$$V^2(t,T_{i-1},T_i) =$$

Inflation-indexed caplets: the JY model (cont'd)

$$= \frac{\sigma_n^2}{2a_n^3}(1 - e^{-a_n \Delta T_i})^2[1 - e^{-2a_n(T_{i-1}-t)}] + \frac{\sigma_r^2}{2a_r^3}(1 - e^{-a_r \Delta T_i})^2[1 - e^{-2a_r(T_{i-1}-t)}]$$

$$- 2\rho_{n,r}\frac{\sigma_n \sigma_r}{a_n a_r(a_n + a_r)}(1 - e^{-a_n \Delta T_i})(1 - e^{-a_r \Delta T_i})[1 - e^{-(a_n+a_r)(T_{i-1}-t)}]$$

$$+ \sigma_I^2 \Delta T_i + \frac{\sigma_n^2}{a_n^2}\left[\Delta T_i + \frac{2}{a_n}e^{-a_n \Delta T_i} - \frac{1}{2a_n}e^{-2a_n \Delta T_i} - \frac{3}{2a_n}\right]$$

$$+ \frac{\sigma_r^2}{a_r^2}\left[\Delta T_i + \frac{2}{a_r}e^{-a_r \Delta T_i} - \frac{1}{2a_r}e^{-2a_r \Delta T_i} - \frac{3}{2a_r}\right]$$

$$- 2\rho_{n,r}\frac{\sigma_n \sigma_r}{a_n a_r}\left[\Delta T_i - \frac{1 - e^{-a_n \Delta T_i}}{a_n} - \frac{1 - e^{-a_r \Delta T_i}}{a_r} + \frac{1 - e^{-(a_n+a_r)\Delta T_i}}{a_n + a_r}\right]$$

$$+ 2\rho_{n,I}\frac{\sigma_n \sigma_I}{a_n}\left[\Delta T_i - \frac{1 - e^{-a_n \Delta T_i}}{a_n}\right] - 2\rho_{r,I}\frac{\sigma_r \sigma_I}{a_r}\left[\Delta T_i - \frac{1 - e^{-a_r \Delta T_i}}{a_r}\right].$$

Inflation-indexed caplets: the JY model (cont'd)

We finally have:

$$
\begin{aligned}
&\textbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) \\
&= \omega N \psi_i P_n(t, T_i) \left[\frac{P_n(t, T_{i-1})}{P_n(t, T_i)} \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} e^{C(t, T_{i-1}, T_i)} \right. \\
&\quad \cdot \Phi\left(\omega \frac{\ln \frac{P_n(t, T_{i-1}) P_r(t, T_i)}{K P_n(t, T_i) P_r(t, T_{i-1})} + C(t, T_{i-1}, T_i) + \frac{1}{2} V^2(t, T_{i-1}, T_i)}{V(t, T_{i-1}, T_i)} \right) \\
&\quad \left. - K \Phi\left(\omega \frac{\ln \frac{P_n(t, T_{i-1}) P_r(t, T_i)}{K P_n(t, T_i) P_r(t, T_{i-1})} + C(t, T_{i-1}, T_i) - \frac{1}{2} V^2(t, T_{i-1}, T_i)}{V(t, T_{i-1}, T_i)} \right) \right],
\end{aligned}
$$

which is clearly of a Black and Scholes type.

Inflation-indexed caplets: a market model

We now try and calculate the IICF price under a market model. To this end, we apply the tower property of conditional expectations to get

$$
\begin{aligned}
&\textbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) \\
&= N\psi_i P_n(t, T_i) E_n^{T_i} \left\{ \frac{E_n^{T_i}\left\{ [\omega(I(T_i) - K I(T_{i-1}))]^+ \,|\mathcal{F}_{T_{i-1}} \right\}}{I(T_{i-1})} \Big| \mathcal{F}_t \right\},
\end{aligned}
$$

where we assume that $I(T_{i-1}) > 0$.

Sticking to a market-model approach, the calculation of the inner expectation depends on whether we model forward rates or the forward CPIs.

We here follow our second market-model approach, since it allows us to derive a simpler formula with less input parameters.

Inflation-indexed caplets: a market model (cont'd)

Assuming again that, under $Q_n^{T_i}$,

$$d\mathcal{I}_i(t) = \sigma_{I,i}\mathcal{I}_i(t)\,dW_i^I(t)$$

and remembering that $I(T_i) = \mathcal{I}_i(T_i)$, we have:

$$\begin{aligned}
&E_n^{T_i}\Big\{\big[\omega(I(T_i) - KI(T_{i-1}))\big]^+ \,|\mathcal{F}_{T_{i-1}}\Big\}\\
&= E_n^{T_i}\Big\{\big[\omega(\mathcal{I}_i(T_i) - KI(T_{i-1}))\big]^+ \,|\mathcal{F}_{T_{i-1}}\Big\}\\
&= \omega\mathcal{I}_i(T_{i-1})\Phi\left(\omega\frac{\ln\frac{\mathcal{I}_i(T_{i-1})}{KI(T_{i-1})} + \frac{1}{2}\sigma_{I,i}^2(T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}}\right)\\
&\quad - \omega KI(T_{i-1})\Phi\left(\omega\frac{\ln\frac{\mathcal{I}_i(T_{i-1})}{KI(T_{i-1})} - \frac{1}{2}\sigma_{I,i}^2(T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}}\right).
\end{aligned}$$

Inflation-indexed caplets: a market model (cont'd)

By definition of $\mathcal{I}_{i-1}$, the IICF price thus becomes

$$\omega N \psi_i P_n(t,T_i) E_n^{T_i} \left\{ \frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} \Phi\left(\omega \frac{\ln \frac{\mathcal{I}_i(T_{i-1})}{K\mathcal{I}_{i-1}(T_{i-1})} + \frac{1}{2}\sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}} \right) \right.$$

$$\left. - K \Phi\left(\omega \frac{\ln \frac{\mathcal{I}_i(T_{i-1})}{K\mathcal{I}_{i-1}(T_{i-1})} - \frac{1}{2}\sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}} \right) \Big| \mathcal{F}_t \right\}.$$

Remembering the dynamics of $\mathcal{I}_{i-1}$ under $Q_n^{T_i}$, and freezing again the drift at its time-t value, we have that under $Q_n^{T_i}$:

$$\ln \frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} \Big| \mathcal{F}_t \sim \mathcal{N}\left(\ln \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} + D_i(t) - \frac{1}{2} V_i^2(t), V_i^2(t) \right).$$

where $V_i^2(t) := (\sigma_{I,i-1}^2 + \sigma_{I,i}^2 - 2\rho_{I,i}\sigma_{I,i-1}\sigma_{I,i})(T_{i-1} - t)$. Therefore,

Inflation-indexed caplets: a market model (cont'd)

$$\mathbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega)$$
$$= \omega N \psi_i P_n(t, T_i) \left[\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} e^{D_i(t)} \Phi\left(\omega \frac{\ln \frac{\mathcal{I}_i(t)}{K \mathcal{I}_{i-1}(t)} + D_i(t) + \frac{1}{2}\mathcal{V}_i^2(t)}{\mathcal{V}_i(t)} \right) \right.$$
$$\left. - K \Phi\left(\omega \frac{\ln \frac{\mathcal{I}_i(t)}{K \mathcal{I}_{i-1}(t)} + D_i(t) - \frac{1}{2}\mathcal{V}_i^2(t)}{\mathcal{V}_i(t)} \right) \right],$$

where $\mathcal{V}_i(t) := \sqrt{V_i^2(t) + \sigma_{I,i}^2 (T_i - T_{i-1})}$, and $\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} = \frac{1+\tau_i F_n(t;T_{i-1},T_i)}{1+\tau_i F_r(t;T_{i-1},T_i)}$.

This price depends on the volatilities of the two forward inflation indices and their correlation, the volatility of nominal forward rates, and the correlations between forward inflation indices and nominal forward rates.

Inflation-indexed caps

An inflation-indexed cap is a stream of inflation-indexed caplets.

An analogous definition holds for an inflation-indexed floor.

Given the set of dates $T_0, T_1, \ldots, T_M$, with $T_0 = 0$, a IICapFloor pays off, at each time T_i, $1, \ldots, M$,

$$N\psi_i \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa \right) \right]^+ ,$$

where κ is the IICapFloor strike, ψ_i are the contract year fractions for the intervals $[T_{i-1}, T_i]$, $1, \ldots, M$, N is the contract nominal value, $\omega = 1$ for a cap and $\omega = -1$ for a floor.

We again set $K := 1 + \kappa$, $\mathcal{T} := \{T_1, \ldots, T_M\}$ and $\Psi := \{\psi_1, \ldots, \psi_M\}$.

Inflation-indexed caps (cont'd)

Sticking to our market model, from the caplet pricing formula we get:

$$\mathbf{IICapFloor}(0,\mathcal{T},\Psi,K,N,\omega) = \omega N \sum_{i=1}^{M} \psi_i P_n(0,T_i)$$

$$\cdot \left[\frac{1+\tau_i F_n(0;T_{i-1},T_i)}{1+\tau_i F_r(0;T_{i-1},T_i)} e^{D_i(0)} \Phi\left(\omega \frac{\ln \frac{1+\tau_i F_n(0;T_{i-1},T_i)}{K[1+\tau_i F_r(0;T_{i-1},T_i)]} + D_i(0) + \frac{1}{2}\mathcal{V}_i^2(0)}{\mathcal{V}_i(0)} \right) \right.$$

$$\left. -K\Phi\left(\omega \frac{\ln \frac{1+\tau_i F_n(0;T_{i-1},T_i)}{K[1+\tau_i F_r(0;T_{i-1},T_i)]} + D_i(0) - \frac{1}{2}\mathcal{V}_i^2(0)}{\mathcal{V}_i(0)} \right) \right].$$

This price depends on the volatilities of forward inflation indices and their correlations, the volatilities of nominal forward rates, and the instantaneous correlations between forward inflation indices and nominal forward rates.

Conclusions

The pricing of inflation-indexed derivatives requires the modelling of both nominal and real rates and of the reference consumer price index.

The foreign-currency analogy allows one to view real rates as the rates in a foreign economy and to treat the CPI as an exchange rate.

Assuming a Gaussian distribution for both instantaneous (nominal and real) rates, as in the Jarrow and Yildirim (2003) model, we have derived analytical formulas for inflation-indexed swaps and caps.

We have also proposed two alternative market-model approaches leading to analytical formulas with easier-to-estimate input parameters.

Our market-model approach allows for extensions based on forward volatility uncertainty in the spirit of Brigo, Mercurio and Rapisarda (2004) or Gatarek (2003).