SMILE AT THE UNCERTAINTY

D. Brigo, F. Mercurio and F. Rapisarda*

Introduction

The success of the Black-Scholes (BS) formula is mainly due to the possibility of synthesizing option prices through a unique parameter, the implied volatility, which is so crucial for traders to be directly quoted in many financial markets. This is because the Black-Scholes formula allows one to immediately convert a volatility into the price at which the related option can be exchanged.

The Black-Scholes model, however, can not be used to price simultaneously all options in a given market. In fact, the assumption of a deterministic underlying-asset volatility leads to constant implied volatilities for any fixed maturity, in contrast with the smile/skew effect commonly observed in practice. Moreover, historical analysis shows that volatilities are indeed stochastic.

Stochastic volatility models, therefore, seem to be a more realistic choice when modelling asset price dynamics for valuing derivative securities. However, only few examples, see Hull and White (1987) or Heston (1993), retain enough analytical tractability so as to be relevant in practice. In general, the calibration to market option prices, and the consequent book re-evaluation, can be extremely burdensome and time consuming.

Stochastic volatility models can also be problematic as far as hedging is concerned: hedging volatility changes is less straightforward than in the Black-Scholes case where we just have one volatility parameter. In general, we can only calculate the sensitivity with respect to the model parameters, which may have an economic meaning but are likely not to have a clear impact on implied volatility surfaces.

The purpose of this paper is to propose a stochastic-volatility model that is analytically tractable as much as Black and Scholes's and for which Vega hedging can be defined in a natural way. The model is based on an uncertain volatility whose random value is drawn, on an infinitesimal future time, from a finite distribution. The model is similar in spirit to (and developed independently from) that of Alexander et al. (2003).

*Product and Business Development, Banca IMI, Corso Matteotti, 6, 20121, Milan, Italy. We are grateful to Aleardo Adotti, head of the Product and Business Development at Banca IMI, for his constant support and encouragement. We are also grateful to Antonio Castagna for disclosing us the secrets of the FX options market and for stimulating discussions. Special thanks go to Lorenzo Bisesti for his fundamental work on the model implementation.

Our uncertain volatility model is equivalent to assuming a number of different possible scenarios for the asset forward volatility, which can therefore be hedged accordingly. Avellaneda *et al.* (1995, 1996) suggested an uncertain volatility model based on the postulate that forward volatility can vary inside a band $[\sigma_{min}, \sigma_{max}]$, thus mapping the pricing problem onto the numerical solution of a nonlinear partial differential equation that yields the value of derivatives under the worst-case volatility scenario. However, choosing a finite number of possible forward volatility states, rather than a full band of possible values, enjoys the same degree of analytical tractability as the original Black–Scholes model. As a direct consequence, prices of exotic claims, even with path-dependent or early-exercise features, are simply mixtures of the corresponding prices in the Black and Scholes model. This renders our model particularly useful in the FX market, where a trader's book typically contains thousands of barrier (or other exotic) options. In fact, we can calculate P&Ls and sensitivities analytically and consistently with smile effects.[1]

The model description

We assume that the asset price dynamics under the risk neutral measure is

$$dS(t) = \begin{cases} S(t)[\mu(t)\,dt + \sigma_0\,dW(t)] & t \in [0, \varepsilon] \\ S(t)[\mu(t)\,dt + \eta(t)\,dW(t)] & t > \varepsilon \end{cases} \tag{1}$$

with $S(0) = S_0 > 0$, and where W is a standard Brownian motion, η is a random variable that is independent of W, σ_0 and ε are positive constants and the risk-neutral drift rate μ is a deterministic function of time. The random variable η takes values in a set of N (given) deterministic functions σ_i with probability λ_i, and $\eta(t)$ denotes its generic value. We thus have:

$$(t \mapsto \eta(t)) = \begin{cases} (t \mapsto \sigma_1(t)) & \text{with probability } \lambda_1 \\ (t \mapsto \sigma_2(t)) & \text{with probability } \lambda_2 \\ \quad\vdots & \qquad\qquad\vdots \\ (t \mapsto \sigma_N(t)) & \text{with probability } \lambda_N \end{cases}$$

where the λ_i are strictly positive and add up to one. The random value of η is assumed to be drawn at time $t = \varepsilon$.[2]

The intuition behind this model is as follows. Our asset price process is nothing but a Black-Scholes geometric Brownian motion where the asset volatility is unknown and one assumes different scenarios for it. The volatility uncertainty applies to an infinitesimal initial time interval with length ε, at the end of which the future volatility value is drawn.

---

[1] Our model can also be viewed as a particular regime-switching model where its coefficients follow very simple Markov chains. Our approach, however, results in a model that is much more tractable than the existing ones in the related literature.

[2] As most stochastic volatility models, our asset price dynamics is directly expressed under a given risk neutral measure. The existence of such a measure, and accordingly the absence of arbitrage, is formally proved in Mercurio (2002).

Therefore, S evolves, for an infinitesimal time, as a geometric Brownian motion with constant volatility σ_0, and then as a geometric Brownian motion with the deterministic volatility $\sigma_i(t)$ drawn at time ε.

We denote by $\mathcal{F}_t^W$ the σ-field generated by W up to time t and by $\mathcal{F}^\eta$ the σ-field associated with η. Since W and σ are independent we take as underlying filtration $\{\mathcal{F}_t = \mathcal{F}_t^W \otimes \mathcal{F}^\eta : t \geq 0\}$.

Setting $\sigma_i(t) = \sigma_0$ for each $t \in [0, \varepsilon]$ and each i, and

$$M(t) := \int_0^t \mu(s)\,ds, \quad V_i(t) := \sqrt{\int_0^t \sigma_i^2(s)\,ds}$$

we have that the density of S at time $t > \varepsilon$ is the following mixture of lognormal densities:

$$p_t(y) = \sum_{i=1}^{N} \lambda_i \frac{1}{yV_i(t)\sqrt{2\pi}} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{S_0} - M(t) + \tfrac{1}{2}V_i^2(t)\right]^2\right\}. \tag{2}$$

In fact, denoting by Q the risk neutral probability, we have

$$Q\{S(t) \leq y\} = \sum_{i=1}^{N} Q\big\{\{S(t) \leq y\} \cap \{\eta(t) = \sigma_i(t)\}\big\} = \sum_{i=1}^{N} \lambda_i\, Q\{S(t) \leq y | \eta(t) = \sigma_i(t)\}.$$

Our model (1) is thus referred to as lognormal-mixture uncertain volatility (LMUV) model.

A direct consequence of (2) is that, under the LMUV model, European option prices are mixtures of Black and Scholes' prices. For instance the arbitrage-free price of a European call with strike K and maturity T is

$$P(0,T)\sum_{i=1}^{N}\lambda_i\left[S_0 e^{M(T)}\Phi\!\left(\frac{\ln\frac{S_0}{K} + M(T) + \frac{1}{2}V_i^2(T)}{V_i(T)}\right) - K\Phi\!\left(\frac{\ln\frac{S_0}{K} + M(T) - \frac{1}{2}V_i^2(T)}{V_i(T)}\right)\right], \tag{3}$$

where $P(t,T)$ denotes the discount factor at time t for maturity T and Φ the standard normal cumulative distribution function.[3]

The LMUV model features the following interesting properties: i) explicit dynamics; ii) explicit marginal densities; iii) explicit formulas for European-style derivatives at the initial time; iv) the analytical tractability at the initial time is extended to all those derivatives which can be explicitly priced under the Black and Scholes paradigm; v) the analytical tractability is preserved after time 0 in that also future option prices can be obtained in an explicit fashion.

The last two properties follow immediately from the model definition. In fact, similarly to what we did for the option price (3), the expectations of functionals of the process (1) can be calculated by conditioning on the possible values of η, thus taking expectations of

[3] Our LMUV model can be viewed as an Hull and White (1987) stochastic volatility model where the volatility is a random variable rather than a stochastic process.

functionals of a geometric Brownian motion. In formulas, any $\mathcal{F}_T$-measurable (and square integrable) payoff V_T at time T has a no-arbitrage price at time $t = 0$ given by

$$V_0 = P(0,T) \sum_{i=1}^{N} \lambda_i E\{V_T \big| \eta = \sigma_i\} = \sum_{i=1}^{N} \lambda_i V_0^{\mathrm{BS}}(\sigma_i) \tag{4}$$

where $V_0^{\mathrm{BS}}(\sigma_i)$ denotes the derivative price under the Black and Scholes model when the asset (time-dependent) volatility is σ_i.

We denote by $r(t)$ the deterministic instantaneous short rate at time t, and assume that the asset S pays a deterministic dividend yield $y(t)$ at time t. For instance, in case S represents an exchange rate, $y(t) = r_f(t)$, where r_f denotes the deterministic instantaneous short rate for the foreign currency. We thus have $\mu(t) = r(t) - y(t)$.

Relationship with the lognormal mixture local volatility model

In terms of marginal densities and prices of European-style derivatives at time 0, our uncertain volatility model is equivalent to the lognormal-mixture local volatility (LMLV) model developed by Brigo and Mercurio (2000, 2001, 2002). More precisely, their LMLV model is the projection of our LMUV counterpart onto the class of local volatility models. We formally prove this in Appendix A. Of the five properties listed above, the LMLV model shares only the first three (i-iii), since in this model only marginal densities are explicitly known.

The substantial difference is that while in the LMLV model randomness in the instantaneous volatility is completely induced instant by instant by randomness in the underlying asset, in the LMUV model randomness acquires a life of its own and, in particular, it is independent of the Brownian motion driving the underlying dynamics. Nevertheless, from the fact that the LMLV model is a sort of projection of the LMUV model follows that the two models share qualitatively several characteristics. For instance, both models decorrelate completely the underlying from its (squared) volatility, see appendix A. Moreover, the future implied volatility curves flatten along the strike dimension both in the LMUV model, after time ϵ and conditional on any chosen scenario, and in the LMLV model (even though not completely, see Brigo (2002) for a numerical example). We will also see below that the two models imply quite similar barrier option prices.

Properties iv) and v) above are important since they allow to transfer the Black-Scholes technology to the LMUV model for all path dependent payoffs, contrary to the LMLV case where numerical methods are needed. If one believed in the LMLV model, hedging would be easier and simply be given by delta hedging, with delta known analytically. This is related to the market completeness of local volatility models and is seemingly in favor of the LMLV formulation. However, this is more a theoretical advantage since financial markets are widely recognized to be incomplete in practice, so that the market incompleteness of the LMUV model, due to the stochastic behavior of the asset volatility, is not such a serious drawback.

	LMLV	LMUV
Marginal densities	Mixture of lognormals	Mixture of lognormals
Transition densities	Unknown	(Mixture of) lognormals
Market completeness	Yes	No
Hedging	Perfect with Delta	Extra assets needed
European options	Mixture of BS prices	Mixture of BS prices
Exotics:	Numerics needed	Mixture of BS-like prices
Corr(S_t,σ_t^2)	0	0

Table 1: The LMUV model and its LMLV projection (σ_t denotes the model instantaneous volatility).

Finally, we notice that since the two models have the same marginal distributions at time 0, calibration to European (plain-vanilla) calls or put is identical under both formulations. When one calibrates the LMUV, one is also implicitly calibrating the LMLV. Calibrating to path dependent options, however, would ideally result in two different sets of parameters for the two models. The relationship between the two models is summarized in Table 1.

Calibration to FX volatility data

Brigo and Mercurio (2001) proved that, for each given maturity T, the option price (3) leads to implied volatility curves, which have a minimum at the at-the-money forward strike $K = S_0 \exp(M(T))$. This renders the LMUV model (1) particularly suitable for calibration to smile-shaped implied volatilities. As a confirmation of this statement, we consider the EUR/USD implied volatilities as of 12 April 2002, which are reported in Table 2. Notice that FX volatilities are typically quoted in terms of deltas rather than strikes.

In our calibration procedure, we parameterize the functions σ_i (actually the corresponding V_i) through Nelson and Siegel's (1987) functions by setting

$$
\begin{aligned}
V_i(t) &= f_i^{NS}(t)\sqrt{t} \\
f_i^{NS}(t) &= a_i + b_i\,(1 - e^{-t/\tau_i})\frac{\tau_i}{t} + c_i\,e^{-t/\tau_i}
\end{aligned}
\tag{5}
$$

We set $N = 3$. By minimizing the mean square error between theoretical and market prices, we obtain the model implied volatilities shown in Table 3.

The absolute differences between model and market volatilities are reported in Table 4. Such differences could be sensibly reduced by choosing a less parsimonious parametrization, like for instance a piece-wise constant one. In this section, however, we just wanted to show the fitting potential of the LMUV model, rather than hinting at possible efficient choices for the "volatility" functions σ_i.

	25Δ	50Δ	75Δ
1W	9.83%	9.45%	9.63%
2W	9.76%	9.40%	9.61%
1M	9.66%	9.25%	9.41%
2M	9.76%	9.40%	9.61%
3M	10.16%	9.85%	10.11%
6M	10.66%	10.40%	10.71%
9M	10.90%	10.65%	11.98%
1Y	10.99%	10.75%	11.09%
2Y	11.12%	10.85%	11.17%

Table 2: EUR/USD implied volatilities as of 12 April 2002.

	25Δ	50Δ	75Δ
1W	9.55%	9.09%	9.55%
2W	9.66%	9.20%	9.67%
1M	9.76%	9.30%	9.76%
2M	10.06%	9.59%	10.07%
3M	10.32%	9.82%	10.35%
6M	10.84%	10.31%	10.89%
9M	11.12%	10.58%	11.19%
1Y	11.27%	10.73%	11.35%
2Y	11.39%	10.90%	11.53%

Table 3: Calibrated volatilities obtained under the parametrization (5).

The analytical pricing of barrier options

We have seen before that our uncertain volatility model implies closed form prices for all those payoffs which can be analytically priced in the Black and Scholes framework with time dependent coefficients.[4] Pricing analytically path-dependent derivatives may be fundamental in markets, like the FX one, where a typical trader's book contains thousands of non-vanilla instruments.

The assumption of a geometric Brownian motion with time-dependent coefficients can render the pricing of these derivatives less straightforward than in the constant-parameters case. However, it is usually possible to come up with analytical approximations, like in the case of the barrier option formulas developed by Lo and Lee (2001). Following their approach, and remembering (4), we obtain, for instance, that the price at time $t = 0$ of an

[4]Using time-dependent coefficients is essential for retrieving the correct term structures of zero-coupon rates or at-the-money volatilities.

	25Δ	50Δ	75Δ
1W	-0.28%	-0.36%	-0.08%
2W	-0.10%	-0.20%	0.06%
1M	0.10%	0.05%	0.35%
2M	0.30%	0.19%	0.46%
3M	0.16%	-0.03%	0.24%
6M	0.18%	-0.09%	0.18%
9M	0.22%	-0.07%	0.21%
1Y	0.28%	-0.02%	0.26%
2Y	0.27%	0.05%	0.36%

Table 4: Absolute differences between calibrated volatilities and market volatilities.

up-and-out call (UOC) with barrier level $H > S_0$, strike K and maturity T is approximately

$$
\begin{aligned}
\mathrm{UOC}_0 = & 1_{\{K<H\}} \sum_{i=1}^{N} \lambda_i \Bigg\{ S_0 e^{c_1+c_2+c_3} \left[\Phi\left(\frac{\ln \frac{S_0}{K} + c_1 + 2c_2}{\sqrt{2c_2}} \right) - \Phi\left(\frac{\ln \frac{S_0}{H} + c_1 + 2c_2}{\sqrt{2c_2}} \right) \right] \\
& - K e^{c_3} \left[\Phi\left(\frac{\ln \frac{S_0}{K} + c_1}{\sqrt{2c_2}} \right) - \Phi\left(\frac{\ln \frac{S_0}{H} + c_1}{\sqrt{2c_2}} \right) \right] - H e^{c_3 + (\beta-1)(\ln \frac{S_0}{H} + c_1) + (\beta-1)^2 c_2} \\
& \cdot \left[\Phi\left(\frac{\ln \frac{S_0}{H} + c_1 + 2(\beta-1)c_2}{\sqrt{2c_2}} \right) - \Phi\left(\frac{\ln \frac{S_0 K}{H^2} + c_1 + 2(\beta-1)c_2}{\sqrt{2c_2}} \right) \right] \\
& + K e^{c_3 + \beta(\ln \frac{S_0}{H} + c_1) + \beta^2 c_2} \left[\Phi\left(\frac{\ln \frac{S_0}{H} + c_1 + 2\beta c_2}{\sqrt{2c_2}} \right) - \Phi\left(\frac{\ln \frac{S_0 K}{H^2} + c_1 + 2\beta c_2}{\sqrt{2c_2}} \right) \right] \Bigg\},
\end{aligned}
\tag{6}
$$

where $1_{\{A\}}$ denotes the indicator function of the set A, and

$$
\begin{aligned}
& c_1 = c_1^i := R(0,T) - Y(0,T) - \tfrac{1}{2} V_i^2(0,T) \\
& c_2 = c_2^i := \tfrac{1}{2} V_i^2(0,T) \\
& c_3 := -R(0,T) \\
& \beta = \beta^i := -2 \frac{\int_0^T [R(t,T) - Y(t,T) - \frac{1}{2} V_i^2(t,T)] V_i^2(t,T)\, dt}{\int_0^T V_i^4(t,T)\, dt} \\
& R(t,T) := \int_t^T r(s)\, ds \\
& Y(t,T) := \int_t^T y(s)\, ds \\
& V_i^2(t,T) := \int_t^T \sigma_i^2(s)\, ds
\end{aligned}
$$

A complete list of formulas, including those for windows, digital and double barriers can be found in Rapisarda (2003).

Examples of barrier option prices under the LMUV model, calibrated to the market data of Table 2, are reported in Table 5, where they are compared with those for the LMLV counterpart and those obtained under a Black and Scholes model with a time-dependent volatility that is stripped from the at-the-money quotes.[5]

As we can infer from this table, the LMUV and LMLV barrier option prices are typically close to each other. This can be motivated by the fact that barrier options are path-dependent derivatives only in a weak sense. However, the purpose of our example is not to suggest a further analogy between the LMUV and LMLV models, but rather to show the possible differences, in terms of option prices, between our model and what is commonly used in the market, at least for books re-evaluations.

Type	LMUV	LMLV	BS
UOC(T=3M,K=1,H=1.05)	28	29	34
UOC(T=6M,K=1,H=1.08)	47	48	57
UOC(T=9M,K=1,H=1.10)	55	57	68
DOC(T=3M,K=1.02,H=0.95)	98	96	99
DOC(T=6M,K=1.07,H=0.98)	67	65	61
DOC(T=9M,K=1.10,H=0.90)	70	68	61

Table 5: Barrier option prices in basis points (we set $S_0 = 1$). DOC denotes the price of a down and out call.

A simple extension of the LMUV model

The presence in the market of skew-shaped implied volatility curves pushes us to consider an extension of model (1) that accounts for possible asymmetries.

The simplest way to do so is by shifting the geometric Brownian motion followed by S under each possible scenario. We thus come up with the following asset price dynamics under the risk neutral measure:

$$dS(t) = \begin{cases} S(t)[\mu(t)\,dt + \sigma_0\,dW(t)] & t \in [0, \varepsilon] \\ \mu(t)S(t)\,dt + \psi(t)[S(t) - \alpha e^{M(t)}]\,dW(t) & t > \varepsilon \end{cases} \tag{7}$$

where (ψ, α) is a random pair that is independent of W and takes values in a set of N

[5]The LMLV barrier option price is obtained through Monte Carlo simulation of the dynamics (12) based on 50000 paths with simulation timestep roughly equal to one seventh of a day. We checked that the resulting discretization bias is much smaller than the statistical uncertainty achieved on the price estimation; the latter amounts at most to 1 bp.

(given) pairs of deterministic functions and real constants:

$$
(t \mapsto \psi(t), \alpha) = \begin{cases} (t \mapsto \sigma_1(t), \alpha_1) & \text{with probability } \lambda_1 \\ (t \mapsto \sigma_2(t), \alpha_2) & \text{with probability } \lambda_2 \\ \quad\vdots & \quad\quad\vdots \\ (t \mapsto \sigma_N(t), \alpha_N) & \text{with probability } \lambda_N \end{cases}
$$

The random value of (ψ, α) is again drawn at time $t = \varepsilon$.

Also in this case, we have a clear interpretation for the asset price dynamics: the extended model (7) is a displaced Black-Scholes model where both the asset volatility and the displacement are unknown. Accordingly, one assumes different (joint) scenarios for them.

This extension has the same advantages as the LMUV model: i) explicit marginal density (mixture of displaced lognormal densities); ii) explicit option prices (mixtures of displaced Black-Scholes prices); iii) explicit transitions densities; iv) explicit (approximated) prices for barrier options.[6] In addition, the extended model can lead to a nice fitting to skew-shaped implied volatility curves and surfaces, since the displacement parameters induce model implied volatilities whose minimum is shifted with respect to the at-the-money forward price.

In real financial markets, we can easily observe implied volatilities with different skews for different maturities. Our displacement parameters α_i are, however, associated to the possible scenarios rather than to the quoted maturities. This prevents us from achieving (almost) perfect calibrations to general skew-shaped surfaces. To this end, we must resort to a more a general extension, which we illustrate in the following.

A general extension of the LMUV model

In order to accommodate general implied volatility skews, we would want the different shifts in the asset price distribution to be associated to the market maturities rather than to the different scenarios one is assuming. A possible way to tackle this issue is by introducing a stochasticity also in the interest rates evolution. In fact, under deterministic rates, the no-arbitrage requirement that the risk neutral expectation of a future asset price be the current forward price somehow limits the possibility of asymmetries in the implied volatility curves. Under stochastic rates, instead, such a constraint applies to expectations under the corresponding forward measure, thus granting us some freedom when modelling the asset price dynamics under the risk neutral measure.

We consider a very simple model for the interest rates (stochastic) evolution, namely an uncertain short-rate model. We assume that at time $t = \varepsilon$ one also draws a deterministic short-rate process and a deterministic dividend yield. Therefore, the new asset price

[6]Barrier option prices under a displaced geometric Brownian motion can be derived under similar, but less straightforward, arguments as those used by Lo and Lee (2001).

dynamics under the risk neutral measure is

$$dS(t) = \begin{cases} S(t)[(r(t) - y(t))\,dt + \sigma_0\,dW(t)] & t \in [0, \varepsilon] \\ S(t)[(\rho(t) - q(t))\,dt + \chi(t)\,dW(t)] & t > \varepsilon \end{cases} \tag{8}$$

where (ρ, q, χ) is a random triplet that is independent of W and takes values in the set of N (given) triplets of deterministic functions:

$$(t \mapsto (\rho(t), q(t), \chi(t))) = \begin{cases} (t \mapsto (r_1(t), y_1(t), \sigma_1(t))) & \text{with probability } \lambda_1 \\ (t \mapsto (r_2(t), y_2(t), \sigma_2(t))) & \text{with probability } \lambda_2 \\ \qquad\vdots & \qquad\vdots \\ (t \mapsto (r_N(t), y_N(t), \sigma_N(t))) & \text{with probability } \lambda_N \end{cases}$$

The random value of (ρ, q, χ) is again drawn at time $t = \varepsilon$.[7]

Also in this more general case, we have a clear interpretation. The extended model is a Black-Scholes model where the asset volatility, the risk free rate and the dividend yield are unknown, and one assumes different (joint) scenarios for them.

The general LMUV model has the same advantages as the LMUV model: i) explicit marginal density (mixture of lognormals with different means); ii) explicit option prices (mixtures of Black-Scholes prices); iii) explicit transitions densities, and hence future option prices; iv) explicit (approximated) prices for barrier options and other exotics. In addition, the extended model leads to a potentially perfect fitting to any (smile-shaped or skew-shaped) implied volatility curves and surfaces.

When calculating derivatives prices under (8), we must remember we are now dealing with stochastic rates and that the discount factor $P(0, T)$ can not be taken out from the risk neutral expectation as we did in (4). In this case, we instead have

$$V_0 = \sum_{i=1}^{N} \lambda_i e^{-\int_0^T r_i(u)\,du} E\big\{V_T \big| (\rho, q, \chi) = (r_i, y_i, \sigma_i)\big\} = \sum_{i=1}^{N} \lambda_i V_0^{\mathrm{BS}}(r_i, y_i, \sigma_i) \tag{9}$$

where $V_0^{\mathrm{BS}}(r_i, y_i, \sigma_i)$ denotes the derivative price under the Black and Scholes model when the (time-dependent) risk-free rate is r_i, the (time-dependent) dividend yield is y_i and the (time-dependent) asset volatility is σ_i, and where we set $r_i(t) = r(t)$, $y_i(t) = y(t)$ and $\sigma_i(t) = \sigma_0$ for $t \in [0, \varepsilon]$ and each i.[8]

The general LMUV model: application to the FX options market

We now consider a second example of calibration to real market FX data. In this case, we remind that the dividend yield coincides with the foreign instantaneous short rate.

[7] One can of course consider the more general random triplet which assigns a non-zero probability to a generic $t \mapsto (r_i(t), y_j(t), \sigma_k(t))$, with a further increase in the number of model parameters (more λ's).

[8] If one wishes to calibrate the uncertain interest-rate model to the market interest rate curve, one has to impose the constraint (10) below.

Since we want to exactly fit both the domestic and foreign zero coupon curves at the initial time, the following no-arbitrage constraints must be imposed:

- Exact fitting to the domestic zero-coupon curve:

$$\sum_{i=1}^{N} \lambda_i e^{-\int_0^t r_i(u)\,du} = P(0,t) \tag{10}$$

- Exact fitting to the foreign zero-coupon curve:

$$\sum_{i=1}^{N} \lambda_i e^{-\int_0^t q_i(u)\,du} = P^f(0,t) \tag{11}$$

  where $P^f(0,t)$ denotes the discount factor for the maturity t in the foreign interest rate market.[9]

The market volatility surface we consider is shown in Figure 1, for Deltas ranging from 10% to 90% and for the same maturities as in Table 2.

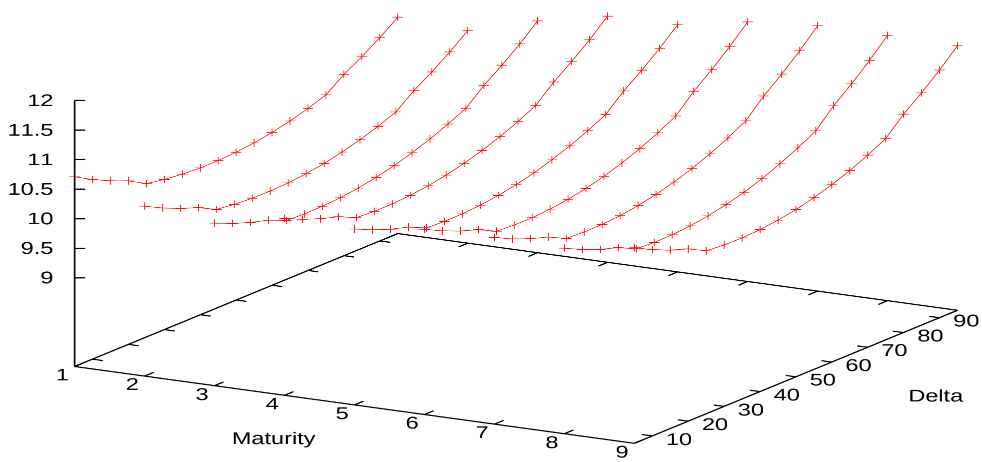

Figure 1: EUR/USD implied volatilities as of 4 February 2003.

For the calibration we decided to resort to a non parametric (cross-section) estimate of functions q and χ, thus considering a deterministic domestic rate ρ. In fact, in turns out that setting $N = 2$ and assuming uncertain asset volatility and foreign rates is sufficient

[9]One may wonder whether it is correct to use the same λ's both for the domestic and foreign risk-neutral measures. However, we can easily prove that such probabilities do not change when changing measure essentially due to the independence between W and (ρ, q, χ).

for achieving a perfect calibration to the three main volatility quotes (25Δ, 50Δ, 75Δ) for all maturities simultaneously.

The relative differences between model and market implied volatilities are shown in Figure 2, where the null difference for the three main Deltas is explicitly highlighted.

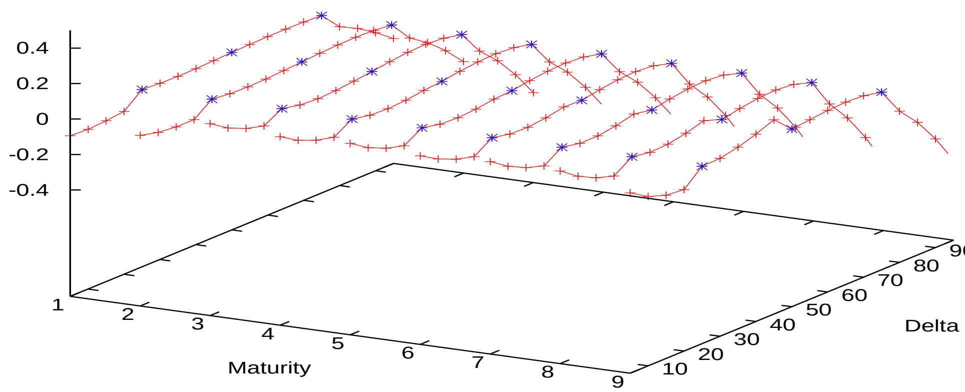

Figure 2: Differences between calibrated volatilities and market volatilities.

Conclusions

We have proposed a simple stochastic volatility model that possesses the same analytical tractability of Black and Scholes's. Explicit formulas can in fact be derived both for European-style and path-dependent derivatives, and both at a current or any future times.

The model is based on an uncertain asset-price volatility, which is drawn (and hence known) at an infinitesimal future time. The assumption of deterministic interest rates leads to implied volatilities with a minimum at the at-the-money forward prices, thus rendering the model suitable for calibration to smile-shaped curves and surfaces.

Asymmetric structures or skew-shaped implied volatilities can be calibrated by assuming an uncertainty also in the interest rates evolution. Our experience in the FX market, for instance, is that we can typically obtain an exact fitting to the main volatility quotes, so that a claim sensitivities can be calculated by shifting one pillar at a time and then re-calibrating.

Having explicit transition densities, the model allows for diagnostics on the future implied volatility surfaces. This can be carried out in a twofold manner: i) we can condition on a future realization of the underlying asset price and volatility thus obtaining a flat implied volatility for each maturity, or ii) we can calculate today's price of forward-start options to infer the forward volatility to be plugged into the Black and Scholes formula to match the model price.

An even more general uncertain volatility model can be considered by assuming subsequent draws of the future asset-price volatility, which will be deterministic between a draw and the next. Such an extension is considered in Mercurio (2002) and is also similar in spirit to that of Alexander et al. (2003). This more general model has the advantage of implying different numbers of density mixtures for different maturities, which typically leads to a better simultaneous calibration to short- and long-dated options when using parsimonious parameterizations. However, exotic option prices are combinations of a number of Black and Scholes prices which grows exponentially with the number of future draws. This is the reason why, in this article, we decided to stick to a simpler formulation.

In its basic formulation with common drifts, the LMUV model originates trough projection the earlier LMLV model (Brigo and Mercurio (2000)). The similarities and differences between the two approaches, with related advantages and disadvantages, have been pointed out. In particular, the wider extension of the Black and Scholes technology achieved with the LMUV has been remarked.

We finally mention that an uncertain volatility can also be considered in a Libor market model framework. In this case, we just have to be careful in specifying each forward rate dynamics under the corresponding forward measure and to treat properly measure changes involving discrete random variables. This is the subject of an ongoing research.

Appendix A: the LMLV model as projection of the LMUV model[10]

If the deterministic functions $\sigma_1, \ldots, \sigma_N$, extended to the interval $[0, \varepsilon]$ where they take the common value σ_0, are also continuous and bounded from below by positive constants, Brigo and Mercurio (2000) proved that the SDE

$$dS(t) = \mu(t)S(t)\,dt + \nu(t, S(t))S(t)\,dW(t) \tag{12}$$

where

$$\nu(t,y) = \sqrt{\frac{\sum_{i=1}^N \lambda_i \sigma_i^2(t) \frac{1}{V_i(t)} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln \frac{y}{S_0} - M(t) + \frac{1}{2}V_i^2(t) \right]^2 \right\}}{\sum_{i=1}^N \lambda_i \frac{1}{V_i(t)} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln \frac{y}{S_0} - M(t) + \frac{1}{2}V_i^2(t) \right]^2 \right\}}}$$

for $(t, y) > (0, 0)$ and $\nu(0, S_0) = \sigma_0$, has a unique strong solution whose marginal density is the lognormal mixture in (2).

Proposition. Under the previous assumptions on the functions σ_i, the LMLV model is the projection of the LMUV model onto the class of local volatility models, in that (see Derman and Kani, 1998)

$$\nu^2(T, K) = E[\eta^2(T) | S(T) = K]$$

[10]We thank Marco Avellaneda for pointing out to us this projection result.

Proof. The equality follows from the definitions of $\eta(t)$ and $\nu(t,y)$ and a simple application of the Bayes rule. In fact,

$$\begin{aligned} E[\eta^2(T)|S(T)=K] &= \sum_{i=1}^{N} \sigma_i^2(T)\, Q\{\eta=\sigma_i|S(T)=K\} \\ &= \sum_{i=1}^{N} \frac{Q\{S(T)=K|\eta=\sigma_i\}\, Q\{\eta=\sigma_i\}}{\sum_{j=1}^{N} Q\{S(T)=K|\eta=\sigma_j\}\, Q\{\eta=\sigma_j\}} \end{aligned}$$

from which we can conclude by noting that $Q\{S(T)=K|\eta=\sigma_i\}$ is the lognormal density corresponding to $V_i(T)$ and calculated in K. □

A further analogy between the LMUV and LMLV models concerns the correlation between the asset price and its (instantaneous or average) squared volatility, which is null in both cases:

$$\begin{aligned} &\mathrm{Corr}\big(\nu^2(t,S(t)),S(t)\big) = \mathrm{Corr}\big(\eta^2(t),S(t)\big) = 0 \\ &\mathrm{Corr}\left(S(t), \int_0^t \eta^2(u)du\right) = \mathrm{Corr}\left(S(t), \int_0^t \nu^2(u,S(u))du\right) = 0. \end{aligned}$$

While in the uncertain volatility case this decorrelation property follows almost by construction from independence of η and W, under the LMLV model it is rather counterintuitive and less straightforward, since in local volatility models volatility is a deterministic function of S_t itself and is thus much more difficult to decorrelate.[11]

References

[1] Alexander, C., Brintalos, G., and Nogueira, L. (2003) Short and Long Term Smile Effects: The Binomial Normal Mixture Diffusion Model. ISMA Centre working paper.

[2] Avellaneda, M., Levy, A. and Parás, A. (1995) Pricing and hedging derivative securities in markets with uncertain volatilities *Applied Mathematical Finance* 2, 73–88.

[3] Avellaneda, M. and Parás, A. (1996) Managing the Volatility Risk of Portfolios of Derivative Securities: the Lagrangian Uncertain Volatility Model *Applied Mathematical Finance* 3, 21–52.

[4] Black, F. and Scholes, M. (1973) The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* 81, 637-659.

[5] Brigo, D. (2002) The general mixture-diffusion SDE and its relationship with an uncertain-volatility option model with volatility-asset decorrelation, Banca IMI internal report. Available at `http://www.damianobrigo.it/sdesmixtures.pdf`

[11] A formal proof of the related result is in Brigo (2002).

[6] Brigo, D. and Mercurio, F. (2000). A mixed-up smile. *Risk* September, 123-126.

[7] Brigo, D. and Mercurio, F. (2001) Displaced and Mixture Diffusions for Analytically-Tractable Smile Models. In *Mathematical Finance - Bachelier Congress 2000*, Geman, H., Madan, D.B., Pliska, S.R., Vorst, A.C.F., eds. Springer Finance, Springer, Berlin, 151-174.

[8] Brigo, D., and Mercurio, F., (2002) Lognormal-Mixture Dynamics and Calibration to Market Volatility Smiles, International Journal of Theoretical & Applied Finance 5, 427-446.

[9] Heston, S. (1993) A Closed Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *Review of Financial Studies* 6, 327-343.

[10] Hull, J. and White, A. (1987) The Pricing of Options on Assets with Stochastic Volatilities. *Journal of Financial and Quantitative Analysis* 3, 281-300.

[11] Lo, C.F. and Lee, H.C. (2001) Single Barrier Options with Time-Dependent Parameters. *Wilmott*, Aug 2001.

[12] Mercurio, F. (2002) A multi-stage uncertain-volatility model. Banca IMI internal report. Available at `http://www.fabiomercurio.it/UncertainVol.pdf`

[13] Nelson C.R. and Siegel, A.F. (1987) Parsimonious modelling of yield curves. *Journal of Business* 60(4), 473-489.

[14] Rapisarda, F. (2003) Pricing barriers on underlyings with time-dependent parameters. Banca IMI internal report. Available at `http://it.geocities.com/rapix/TimeDependentBarriers.pdf`