PRICING INFLATION-INDEXED OPTIONS WITH STOCHASTIC VOLATILITY

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Abstract. In order to recover smile-consistent prices for inflation-indexed caps and floors, we propose a stochastic volatility model for forward consumer price indices, with volatility dynamics as in Heston (1993). Closed-form formulas for inflation-indexed caplets and floorlets based on the Carr and Madan (1998) Fourier transform approach are then derived. An example of calibration to market data and numerical details concerning our pricing procedure are finally given.

1. Introduction

Inflation-indexed (II) derivatives are designed to protect investors against variations in the purchasing power of their currency. Besides II bonds, which have been issued by governments since the beginning of the 80's, there is nowadays an increasing interest for derivatives such as II zero-coupon and II year-on-year swaps. II swaps are swaps where, on each payment date, a financial institution pays (receives) the inflation rate over a predefined period in exchange for a payment based on a fixed rate.

II derivatives with non-linear payoffs are still rather illiquid, with the possible exception of those embedded in II bond issues. The market for II options, however, is expected to grow in the near future and matrices of prices for different strikes and maturities are already provided by brokers.

The pricing of II derivatives is typically addressed by resorting to a foreign-currency analogy. In fact, one can convert nominal values into real ones simply by dividing by the current value of the reference Consumer Price Index (CPI), pretty much as one does when switching from one currency to another. Nominal and real quantities, therefore, can play the role of "domestic" and "foreign" quantities, respectively, and the CPI can be interpreted as the exchange rate between the nominal and the real economies.[1]

The most relevant application of the foreign-currency analogy is due to Jarrow and Yildrim (2003), who modelled the dynamics of the CPI along with nominal and real rates, both assumed to follow a one-factor Gaussian process in the HJM framework. An alternative approach has been independently proposed by Kazziha (1999), Belgrade et al. (2004) and Mercurio (2005), who considered a market model whose underlying variables are forward CPI's evolving as driftless geometric Brownian motions under associated forward measures. We refer to their model as

Date: First Version January 2005. This Version 17th August 2005.

[1]The interest rates in this real economy must be intended as the real rates we can lock in by suitably trading in inflation swaps. The true real rates, however, will be only known at the end of the corresponding periods.

to Lognormal Forward CPI Model (LFCPIM). In this article, we will extend the LFCPIM by introducing stochastic volatility as in Heston (1993).

The purpose of this article is the derivation of closed-form formulas for II caps and floors under the considered market model with stochastic volatility. We will see that an exact formula, in terms of a one-dimensional integration, can be derived under mild assumptions on the model coefficients. The general case will be dealt with an appendix, where ad-hoc approximations will be explicitly provided. The article is completed with an example of calibration to market data.

2. Definitions and Motivations

Let us consider a tenor structure $\{T_0 = 0, T_1, \ldots, T_M\}$ with corresponding years fractions $\{\tau_1, \ldots, \tau_M\}$. Defining inflation as the percentage increment of the CPI over the related time interval, a II cap starting today, maturing at T_M and with strike K is a contract paying out at each time T_i, $i = 1, \ldots, M$, the quantity[2]

$$\left(\frac{I(T_i) - I(T_{i-1})}{I(T_{i-1})} - K\right)^+ = \left(\frac{\mathcal{I}_i(T_i)}{\mathcal{I}_{i-1}(T_{i-1})} - K - 1\right)^+ \tag{1}$$

where we denote by $\mathcal{I}_i(t)$ the T_i-forward CPI at time t, which is defined by

$$\mathcal{I}_i(t) := \frac{I(t)P_r(t, T_i)}{P(t, T_i)} \tag{2}$$

with $P(\cdot, \cdot)$ and $P_r(\cdot, \cdot)$ denoting discount factors in the nominal and real economies, respectively. The total payoff of a cap is then given by the sum over i of single cash flows (1), which are commonly referred to as caplets.

The price of the II caplet that pays (1) at time T_i is denoted by Cplt_i, whereas the price of the corresponding floorlet (a put on the inflation rate) is denoted by Flrt_i.

Representing the payoff (1) in terms of forward quantities has the advantage that forward CPI's are, by definition, price processes of tradable securities in the nominal market, thus being more meaningful than the CPI itself, which is merely treated as an exchange rate. In fact, a T_i-forward CPI is a martingale under the corresponding T_i-forward measure, so that its dynamics, under such a measure, is fully specified by its volatility process.[3]

Assuming that each forward CPI follows a driftless geometric Brownian motion under the corresponding forward measure, Kazziha (1999), Belgrade et al. (2004) and Mercurio (2005) derived a Black-Scholes-like formula for the i-th caplet. Such a formula, which is based on deterministic forward-CPI's volatilities, can be safely used to price II caps with different maturities but same strike. When simultaneously pricing caps with different strikes, however, the natural question arises whether one should include some kind of smile effect, since this is a consolidated practice in all developed options markets.

To address such an issue, let us focus on the first time interval. As at time T_0, $\mathcal{I}_0$ is known and equal to $I(0)$, and the first caplet is in fact a plain-vanilla call on $\mathcal{I}_1$ which, by definition, is a martingale under the T_1-forward measure. Assuming

[2]We assume unit notional value.

[3]The forward measures we consider in this article are to be intended as measures in the nominal economy.

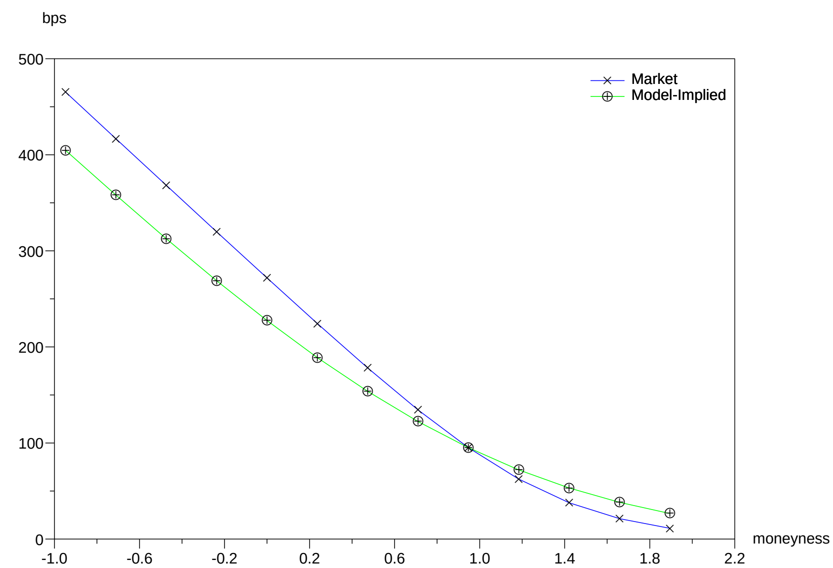

FIGURE 1. Market prices versus model prices with ATM implied volatility for one-year caplets. Moneyness is defined as: $K/(\mathcal{I}_1(0)/I(0)-1)$.

the following lognormal-type dynamics for $\mathcal{I}_1$

$$d\mathcal{I}_1(t) = \sigma_1 \mathcal{I}_1(t)\, dZ_1^I(t),$$

where σ_1 is a positive constant and Z_1^I is a standard Brownian motion, the price at time 0 of the first caplet is thus given by (see for instance Mercurio, 2005)

(3)

$$P(0,T_1)\left[\frac{\mathcal{I}_1(0)}{I(0)}\Phi\left(\frac{\ln\frac{\mathcal{I}_1(0)}{(1+K)I(0)}+\frac{1}{2}\sigma_1^2T_1}{\sigma_1\sqrt{T_1}}\right)-(1+K)\Phi\left(\frac{\ln\frac{\mathcal{I}_1(0)}{(1+K)I(0)}-\frac{1}{2}\sigma_1^2T_1}{\sigma_1\sqrt{T_1}}\right)\right].$$

With caplet quotes available for a number of strikes, it is straightforward to test whether the II caplet market is flat-smiled or not. To this end, we proceed as follows. We first take the at-the-money (ATM) strike,[4] and calculate the ATM implied volatility by inversion of formula (3). We then use this same volatility to price a set of caplets with different strikes. The results we find are reported in Figure 1, where we consider market data as of November 7th, 2004. It is clear that the ATM implied volatility underestimates out-of-the-money (OTM) prices and overestimate in-the-money (ITM) ones, revealing an implicit skew in the market prices. This simple test, therefore, suggests the need for a model that goes beyond the geometric Brownian motion.

Given the empirical evidence of a stochastic-volatility behavior of interest rates (see for instance Andersen et al., 1997) and given also the dependence of forward CPI's on forward rates, it seems natural to resort to an extension that is based on

[4]Or the closest possible one in case this is not immediately available.

stochastic volatility. The main purpose of this article will then be the derivation of closed-form formulas for II caps and floors in such an extended model.

3. Modelling Forward CPI's with Stochastic Volatility

In order to set up a stochastic-volatility framework, we assume that nominal forward rates F_i's, where

$$F_i(t) := [P(t,T_{i-1})/P(t,T_i) - 1]/\tau_i,$$

are lognormally distributed according to a lognormal LIBOR model (see *e.g.* Brace et al., 1997) with constant volatilities, and that forward CPI's, $\mathcal{I}_i$, have a Heston-like evolution with a common volatility $V(t)$ that follows a mean-reverting square-root process, see Heston (1993). In other words we assume that, under a given *reference measure* $\mathbb{Q}$,

$$\begin{aligned} dF_i(t)/F_i(t) &= (\dots)\,dt + \sigma_i^F\, dZ_i^{\mathbb{Q},F}(t) \\ d\mathcal{I}_i(t)/\mathcal{I}_i(t) &= (\dots)\,dt + \sigma_i^I \sqrt{V(t)}\, dZ_i^{\mathbb{Q},I}(t) \\ dV(t) &= \alpha(\theta - V(t))\,dt + \epsilon\sqrt{V(t)}\, dW^{\mathbb{Q}}(t), \end{aligned} \tag{4}$$

where σ_i^F's, σ_i^I's, α, θ and ϵ are positive constants, $2\alpha\theta > \epsilon$ to ensure positiveness of V, and where we allow for correlations between Brownian motions $Z_i^{\mathbb{Q},F}, Z_i^{\mathbb{Q},I}, W^{\mathbb{Q}}$.

Instead of defining the volatility's dynamics, and parameters α, θ, ϵ, under a risk-neutral measure $\mathbb{Q}^*$ or under the terminal forward measure $\mathbb{Q}^M$, having the bond $P(t,T_M)$ as numeraire, we suggest to choose $\mathbb{Q} = \mathbb{Q}^0$, where $\mathbb{Q}^0$ is the spot LIBOR measure, which is related to the numeraire

$$N(t) = P(t,\beta(t)) \prod_{l=1}^{\beta(t)} [1 + \tau F_l(t)], \quad \beta(t) = T_j \quad \text{if } T_{j-1} < t \le T_j. \tag{5}$$

In fact, if we took $\mathbb{Q}^M$ as reference measure, the volatility's evolution would depend on the choice of the last maturity T_M. Choosing instead a particular risk-free measure, would hide the evaluation issue of the market price of volatility risk. On the other side, the spot LIBOR measure is, to many extents, payoff independent, allowing the valuation of a single payoff (1) irrespective of the number of caplets that follow.

The classical change of measure technique, see Geman et al. (1995), implies that, under $\mathbb{Q}^0$, the forward rates and forward CPI's dynamics are given by

$$\begin{aligned} dF_i(t)/F_i(t) &= \sigma_i^F \left[\sum_{l=\beta(t)+1}^{i} \sigma_l^F \rho_{i,l}^F \frac{\tau_l F_l(t)}{1+\tau_l F_l(t)} dt + dZ_i^{0,F}(t) \right] \\ d\mathcal{I}_i(t)/\mathcal{I}_i(t) &= \sqrt{V(t)}\,\sigma_i^I \left[\sum_{l=\beta(t)+1}^{i} \sigma_l^F \rho_{l,i}^{F,I} \frac{\tau_l F_l(t)}{1+\tau_l F_l(t)} dt + dZ_i^{0,I}(t) \right] \\ dZ_i^{0,F}(t)\, dZ_l^{0,F}(t) &= \rho_{i,l}^F\, dt \\ dZ_i^{0,I}(t)\, dZ_l^{0,F}(t) &= \rho_{l,i}^{F,I}\, dt \end{aligned} \tag{6}$$

while $V(t)$ evolves as in (4), and where the last two equations define the correlation parameters $\rho_{i,l}^F$ and $\rho_{l,i}^{F,I}$.

II caplets and floorlets paying at time T_j are derivatives that depend on $\mathcal{I}_j$ and $\mathcal{I}_{j-1}$. It is, therefore, convenient to express dynamics under the forward measure $\mathbb{Q}^j$, under which, by definition, both F_j and $\mathcal{I}_j$ are martingales.

When we take $P(t,T_j)$ as numeraire, the relevant quantities $\mathcal{I}_j(\cdot)$, $\mathcal{I}_{j-1}(\cdot)$ and $X_j(\cdot) := \ln(\mathcal{I}_j(\cdot)/\mathcal{I}_{j-1}(\cdot))$ satisfy the following SDE's

$$
\begin{aligned}
&d\mathcal{I}_j(t)/\mathcal{I}_j(t) = \sqrt{V(t)}\,\sigma_j^I\,dZ_j^I(t)\\
&d\mathcal{I}_{j-1}(t)/\mathcal{I}_{j-1}(t) = \sqrt{V(t)}\,\sigma_{j-1}^I\left[-\frac{\tau_j F_j(t)}{1+\tau_j F_j(t)}\,\sigma_j^F\,\rho_{j,j-1}^{F,I}dt + dZ_{j-1}^I(t)\right]\\
&dX_j(t) = \left[\frac{V(t)}{2}\big((\sigma_{j-1}^I)^2-(\sigma_j^I)^2\big) + \sqrt{V(t)}\sigma_{j-1}^I\sigma_j^F\,\rho_{j,j-1}^{F,I}\frac{\tau_j F_j(t)}{1+\tau_j F_j(t)}\right]dt\\
&\qquad + \sqrt{V(t)}[\sigma_j^I\,dZ_j^I(t) - \sigma_{j-1}^I\,dZ_{j-1}^I(t)]
\end{aligned}
\tag{7}
$$

while the volatility evolves according to

$$
\begin{aligned}
dV(t) &= \left[\alpha\theta - \epsilon m_j(t)\sqrt{V(t)} - \alpha V(t)\right]dt + \epsilon\sqrt{V(t)}\,dW(t)\\
m_j(t) &= \sum_{l=\beta(t)+1}^{j}\frac{\tau_l F_l(t)}{1+\tau_l F_l(t)}\sigma_l^F\,\rho_l^{F,V}
\end{aligned}
\tag{8}
$$

where $dZ_l^F(t)\,dW(t) = \rho_l^{F,V}dt$, for each l.

We denote by $\rho_{j,l}^I$ the correlation between forward CPI's $\mathcal{I}_j$ and $\mathcal{I}_l$,

$$\rho_{j,l}^I\,dt = dZ_j^I(t)\,dZ_l^I(t),$$

and by $\rho_i^{I,V}$ the correlation between the forward CPI $\mathcal{I}_i$ and the volatility,

$$\rho_i^{I,V}\,dt := dZ_i^I(t)\,dW(t).$$

4. Pricing Formulae

Regardless of the model chosen for the CPI's evolution, the price at time $t \le T_j$ of the j-th caplet, is, under measure $\mathbb{Q}^j$,

$$
\begin{aligned}
\mathrm{Cplt}_j(t,K) &= P(t,T_j)\mathbb{E}_t^j\left(\frac{\mathcal{I}_j(T_j)}{\mathcal{I}_{j-1}(T_{j-1})} - (K+1)\right)^+\\
&= P(t,T_j)\int_{-\infty}^{+\infty}(\mathrm{e}^s - \mathrm{e}^k)^+ q_t^j(s)\,ds
\end{aligned}
\tag{9}
$$

where $k = \ln(K+1)$, $q_t^j(s)ds := \mathbb{Q}^j\left\{\ln\left(\mathcal{I}_j(T_j)/\mathcal{I}_{j-1}(T_{j-1})\right) \in [s, s+ds)|\mathcal{F}_t\right\}$, and $\mathbb{E}_t^j$ denotes expectation under $\mathbb{Q}^j$ conditional on the σ-algebra $\mathcal{F}_t$ generated by the relevant processes up to time t.

The difficulty in the calculation of (9) is that, instead of having a payoff depending on a single asset as in case of standard or forward-start (cliquet) options, here the payoff depends on the ratio of *two different assets, $\mathcal{I}_j$ and $\mathcal{I}_{j-1}$ at two different times*, T_j and T_{j-1}.

Following Carr and Madan (1998), we rewrite the caplet price (9) in terms of its (renormalized) Fourier Transform (FT), obtaining

$$\begin{aligned}\mathrm{Cplt}_j(t,\mathrm{e}^k) &= P(t,T_j)\frac{\mathrm{e}^{-\eta k}}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{-iuk}\frac{\phi_t^j(u-(\eta+1)i)}{(\eta+iu)(\eta+1+iu)}du \\ &= P(t,T_j)\frac{\mathrm{e}^{-\eta k}}{\pi}\mathrm{Re}\int_{0}^{+\infty}\mathrm{e}^{-iuk}\frac{\phi_t^j(u-(\eta+1)i)}{(\eta+iu)(\eta+1+iu)}du\end{aligned} \tag{10}$$

where the only unknown function is the conditional characteristic function $\phi_t^j(\cdot)$ of $\ln(\mathcal{I}_j(T_j)/\mathcal{I}_{j-1}(T_{j-1}))$,

$$\phi_t^j(u) = \mathbb{E}_t^j\left[\mathrm{e}^{iu\ln(\mathcal{I}_j(T_j)/\mathcal{I}_{j-1}(T_{j-1}))}\right],$$

and where $\eta \in \mathbb{R}^+$ is used to ensure L^2-integrability when $k \to -\infty$.

Analogously, the price of a II floorlet is given by, see Appendix B,

$$\begin{aligned}\mathrm{Flrt}_j(t,K) &= P(t,T_j)\mathbb{E}_t^j\left((K+1)-\frac{\mathcal{I}_j(T_j)}{\mathcal{I}_{j-1}(T_{j-1})}\right)^+ \\ &= P(t,T_j)\frac{\mathrm{e}^{\tilde{\eta}k}}{\pi}\mathrm{Re}\int_{0}^{+\infty}\mathrm{e}^{-iuk}\frac{\phi_t^j(u+i(\tilde{\eta}-1))}{(iu-\tilde{\eta}+1)(iu-\tilde{\eta})}du\end{aligned}$$

where $\tilde{\eta} \in (1,+\infty)$ has been introduced, like η in (10), for FT regularization. As explained in Appendix B, it is also possible to compute floorlet prices by a put-call parity that holds between II caplets and floorlets.

The pricing of II caplets and floorlets reduces, therefore, to the calculation of a conditional characteristic function. With an explicit formula for ϕ_t^j, pricing can be quite fast thanks to well established FT techniques, thus allowing a not too expensive calibration procedure.

Let us recall that, by definition of characteristic function and the Markov property, we can write

$$\phi_t^j(u) = H\left(V(t), Y_j(t), Y_{j-1}(t), F_1(t), \ldots, F_j(t); u\right), \tag{11}$$

where H is the solution of a PDE that can be found through Feynman-Kač's theorem.

It is clear that in the general framework of (7), due to the unpleasant presence of drift terms like $\sqrt{V(t)}F_l(t)/(1+\tau_l F_l(t))$, the PDE satisfied by H is rather involved and does not seem to be explicitly solvable but under particular conditions or approximations. As a consequence, in the following, we will restrict our attention to some particular cases where the dependence on forward rates can be eliminated so as to yield an explicit expression for $\phi_t^j(u)$.

The closed-form pricing formulae we will derive are the main result of this paper and will be obtained by solving iteratively two Heston's PDE's with different diffusion coefficients and boundary conditions.

4.1. **Exact Solution for the Uncorrelated Case.** The forward CPI's and volatility's dynamics (7) and (8) can be considerably simplified by assuming that

$$\rho_{i,l}^{F,I} = 0, \text{ for each } i,l = 1,\ldots,M, \tag{12}$$

so that each $\mathcal{I}_j$ is a driftless geometric Brownian motion under every forward measure, and

$$\rho_i^{F,V} = 0, \text{ for each } i = 1,\ldots,M, \tag{13}$$

so as to completely separate the SDE's of forward rates and volatility.

Introducing (12) and (13) is extremely convenient from a computational point of view, since it will enable us to derive an explicit expression for the function H in (11).

Assumption (12) has also an empirical justification. In fact the LFCPIM prices of both year-on-year swaps and caplets are not very sensitive to changes in the correlation between forward rates and forward CPI's. Calibration to cross-sectional data thus likely provides no empirical evidence that the *implied* correlation's value is necessarily different than zero. A similar situation can also be encountered under our stochastic-volatility extension, as we will see below.

Under (12) and (13), the cumbersome terms depending on forward rates disappear from the drifts of SDE's (7) and (8), thus leading to

$$\begin{aligned} dY_j(t) &= -\frac{1}{2}V(t)(\sigma_j^I)^2 dt + \sqrt{V(t)}\,\sigma_j^I(t)\,dZ_j^I(t) \\ dX_j(t) &= \frac{V(t)}{2}\big((\sigma_{j-1}^I)^2 - (\sigma_j^I)^2\big)\,dt + \sqrt{V(t)}[\sigma_j^I\,dZ_j^I(t) - \sigma_{j-1}^I\,dZ_{j-1}^I(t)] \\ dV(t) &= [\alpha\theta - \alpha V(t)]\,dt + \epsilon\sqrt{V(t)}\,dW(t) \end{aligned} \tag{14}$$

where we set $Y_j(t) := \ln(\mathcal{I}_j(t))$.

To produce dynamics (14), we set to zero all correlations that explicitly entered the previous SDEs. However, we still allow for non-zero values for $\rho_{j,l}^I$, namely for the correlation between different forward CPI's, and for $\rho_i^{I,V}$, namely for the correlation between forward CPI's and the volatility. Our simplifying assumptions, therefore, are not so restrictive as far as the pricing of II derivatives is concerned.[5]

To make ϕ_t^j explicit, let us rewrite (11) as

$$\phi_t^j(u) = \mathbb{E}_t^j\left[\mathrm{e}^{iu\left(Y_j(T_j) - Y_{j-1}(T_{j-1})\right)}\right] = \mathbb{E}_t^j\left[\mathrm{e}^{-iuY_{j-1}(T_{j-1})}\mathbb{E}_{T_{j-1}}^j\left(\mathrm{e}^{iuY_j(T_j)}\right)\right] \tag{15}$$

and notice that $\mathbb{E}_{T_{j-1}}^j\left(\mathrm{e}^{iuY_j(T_j)}\right)$ is nothing but the characteristic function of $\ln(\mathcal{I}_j(T_j))$ conditional on $\mathcal{F}_{T_{j-1}}$. From the first and third equations of (14) and solving a Heston-like PDE, as detailed in Appendix A, we have that

$$\mathbb{E}_{T_{j-1}}^j\left(\mathrm{e}^{iuY_j(T_j)}\right) = \exp\left\{A_Y(\tau_j,u) + B_Y(\tau_j,u)V(T_{j-1}) + iuY_j(T_{j-1})\right\}$$

where

$$\begin{aligned} B_Y(s,u) &= \frac{\gamma - b}{2a}\left[\frac{1-\mathrm{e}^{\gamma s}}{1-\frac{b-\gamma}{b+\gamma}\mathrm{e}^{\gamma s}}\right] \\ A_Y(s,u) &= \frac{\alpha\theta(\gamma-b)}{2a}s - \frac{\alpha\theta}{a}\ln\left[\frac{1-\frac{b-\gamma}{b+\gamma}\mathrm{e}^{\gamma s}}{1-\frac{b-\gamma}{b+\gamma}}\right] \end{aligned} \tag{16}$$

and

$$\begin{aligned} a &= \epsilon^2/2, & c &= -\,iu(\sigma_j^I)^2/2 - (\sigma_j^I)^2u^2/2, \\ b &= iu\sigma_j^I\epsilon\rho_j^{I,V} - \alpha, & \gamma &= \sqrt{b^2-4ac}. \end{aligned}$$

[5]When pricing, instead, hybrid products that are based on inflation and interest rates, we should allow for general correlations and resort to some *freezing* approximations as explained in Section 4.2.

As a consequence, since $X_j(T_{j-1}) = Y_j(T_{j-1}) - Y_{j-1}(T_{j-1})$, we can write

$$\phi_t^j(u) = \mathrm{e}^{A_Y(\tau_j,u)}\mathbb{E}_t^j\left[\mathrm{e}^{iuX_j(T_{j-1})+B_Y(\tau_j,u)V(T_{j-1})}\right]. \tag{17}$$

The residual expectation in (17), which coincides with its argument if $t \in [T_{j-1}, T_j)$, is the characteristic function of the couple $(X_j(T_{j-1}), V(T_{j-1}))$ evaluated at point $(u, -iB_Y(\tau_j, u))$, and conditional on $\mathcal{F}_t$. An explicit formula for $\phi_t^j(u)$ can then be obtained by solving a new PDE, which is determined by the second and third equation of (14). Proceeding as before, we finally have

$$\phi_t^j(u) = \exp\{A_Y(\tau_j,u) + A_X(T_{j-1}-t,u)B_X(T_{j-1}-t,u)V(t) + iuX_j(t)\} \tag{18}$$

where

$$\begin{aligned} B_X(\tau,u) =& B_Y(\tau_j,u) + \frac{\bar\gamma - \bar b - 2\bar a B_Y(\tau_j,u)}{2\bar a}\left[\frac{1-\mathrm{e}^{\bar\gamma\tau}}{1 - \frac{2\bar a B_Y(\tau_j,u)+\bar b-\bar\gamma}{2\bar a B_Y(\tau_j,u)+\bar b+\bar\gamma}\mathrm{e}^{\bar\gamma\tau}}\right] \\ A_X(\tau,u) =& \frac{\alpha\theta(\bar\gamma-\bar b)}{2\bar a}\tau - \frac{\alpha\theta}{\bar a}\ln\left[\frac{1 - \frac{2\bar a B_Y(\tau_j,u)+\bar b-\bar\gamma}{2\bar a B_Y(\tau_j,u)+\bar b+\bar\gamma}\mathrm{e}^{\bar\gamma\tau}}{1 - \frac{2\bar a B_Y(\tau_j,u)+\bar b-\bar\gamma}{2\bar a B_Y(\tau_j,u)+\bar b+\bar\gamma}}\right] \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\bar a = \epsilon^2/2, \qquad \bar b = iu\epsilon(\sigma_j^I\rho_j^{I,V} - \sigma_{j-1}^I\rho_{j-1}^{I,V}) - \alpha \\ &\bar c = iu\big((\sigma_{j-1}^I)^2 - (\sigma_j^I)^2\big)/2 - \big((\sigma_{j-1}^I)^2 + (\sigma_j^I)^2 - 2\sigma_j^I\sigma_{j-1}^I\rho_{j,j-1}^I\big)u^2/2 \\ &\bar\gamma = \sqrt{\bar b^2 - 4\bar a\bar c}, \end{aligned} \tag{20}$$

where the case $\sigma_j^I = \sigma_{j-1}^I$ with $\rho_{j,j-1}^I = 1$ has been discarded because it leads to a degenerate deterministic evolution for X_j.

Function (18) gives the exact solution to our pricing problem under (12) and (13). Under non-zero correlation structures, the pricing of caplets is less straightforward. We will see in the following, however, that the general case of dynamics (7) and (8) can be addressed by resorting to suitable approximations.

4.2. **Approximated Dynamics for Non-zero Correlations.** The classical way to handle unpleasant drift terms in the LIBOR market model, see also the first equation in (6), is by freezing them at their time-0 value. Here, we deal with the forward-rates dependent ratios in the drifts of (7) by using a similar technique.

The drift terms that involve forward rates are

$$D_l(t) := \sqrt{V(t)}\frac{F_l(t)}{1+\tau_l F_l(t)}$$

and depend on the volatility, too.

A first way to freeze these terms consists in setting

$$D_l(t) \approx D_l(0),$$

and changing the asymptotic volatility value from θ to

$$\tilde\theta := \theta - \frac{\epsilon}{\alpha}\sum_{l=1}^{j} D_l(0)\tau_l\sigma_l^F\rho_l^{F,V}, \tag{21}$$

which leads to the following (approximated) SDEs for X_j and V

$$\begin{aligned} dX_j(t) \approx & \left[\frac{V(t)}{2}\big((\sigma^I_{j-1})^2 - (\sigma^I_j)^2\big) + D_j(0)\tau_j\sigma^I_{j-1}\sigma^F_j\,\rho^{F,I}_{j,j-1}\right]dt \\ & + \sqrt{V(t)}[\sigma^I_j\,dZ^I_j(t) - \sigma^I_{j-1}\,dZ^I_{j-1}(t)] \\ dV(t) \approx & \alpha(\tilde{\theta} - V(t))\,dt + \epsilon\sqrt{V(t)}\,dW(t). \end{aligned} \tag{22}$$

The dynamics of X_j in (22) differs from that in equation (14) for a constant drift term. Such a term, however, is innocuous and the relevant characteristic functions can still be calculated by means of Appendix A.

A second possibility for a tractable approximation is to set

$$D_l(t) \approx \frac{F_l(t)}{1+\tau_l F_l(t)}\frac{V(t)}{\sqrt{V(t)}} \approx \frac{F_l(0)}{1+\tau_l F_l(0)}\frac{V(t)}{\sqrt{V(0)}} = D_l(0)\frac{V(t)}{V(0)}, \tag{23}$$

where the freezing is done with the purpose of producing a linear term in $V(t)$. This leads to the following (approximated) SDEs for X_j and V

$$\begin{aligned} dX_j(t) \approx & V(t)\left[\frac{1}{2}\big((\sigma^I_{j-1})^2 - (\sigma^I_j)^2\big) + \frac{D_j(0)}{V(0)}\tau_j\sigma^I_{j-1}\sigma^F_j\,\rho^{F,I}_{j,j-1}\right]dt \\ & + \sqrt{V(t)}[\sigma^I_j\,dZ^I_j(t) - \sigma^I_{j-1}\,dZ^I_{j-1}(t)] \\ dV(t) \approx & \bar{\alpha}(\bar{\theta} - V(t))\,dt + \epsilon\sqrt{V(t)}\,dW(t), \end{aligned} \tag{24}$$

where

$$\begin{aligned} \bar{\alpha} &:= \alpha + \frac{\epsilon}{V(0)}\sum_{l=1}^{j} D_l(0)\tau_l\sigma^F_l\,\rho^{F,V}_l \\ \bar{\theta} &:= \alpha\theta/\bar{\alpha} \end{aligned} \tag{25}$$

Also in this second case, the relevant characteristic functions can be calculated according to Appendix A.

Remark 4.1. *For the approximated processes $V(t)$ to be meaningful, we must require $\tilde{\theta} > 0$ in (22) and $\bar{\alpha} > 0$ in (24). Moreover, for the origin to be inaccessible, conditions $2\tilde{\theta}\alpha > \epsilon$ and $2\bar{\theta}\bar{\alpha} > \epsilon$ must be imposed, with the latter that is automatically satisfied since $2\theta\alpha > \epsilon$ by assumption.*

Both approximations (22) and (24) lead to characteristic functions that can be explicitly calculated by applying the result in Appendix A, and analogously to the "uncorrelated" case. Given the similarity with (18), the explicit solutions in these two cases are not reported here for brevity.

4.3. **Comparing the Approximations with the Exact Formula.** To test the goodness of the above approximations (22) and (24), we should perform a Monte Carlo simulation of processes (7) and (8) and compare the analytical caplet prices coming from the approximations with the corresponding Monte Carlo price windows calculated numerically. This procedure, however, is rather cumbersome, since it also requires the joint simulation of all forward rates.

A much quicker test can be conducted by simply comparing the caplet prices implied by the two approximations. In fact, the two freezings can lead to quite different terms when the volatility $V(t)$ deviates from its initial value $V(0)$. Therefore, if the approximations are quite close to each other, we may conclude that they should also be very close to the true model's value. And this indeed turns out to be the case in all the numerical experiments we have analyzed.

The two approximations, moreover, can be compared with the exact price obtained under assumptions (12) and (13). To this end, we report in Figure 2 the percentage differences between prices (18) and the ones implied by the two freezing techniques, where we used EUR market data as of October 7, 2004. We also set: $\rho_i^{F,V} = \rho_{i,l}^{F,I} = \rho_i^{I,V} = -0.2$, $\rho_{i,i-1}^{I} = 1 - 1.5\mathrm{e}^{-0.08(i-2)}$, $\alpha = 0.2$, $\theta = 0.001$, $V(0) = 0.001$, $\epsilon = 0.01$ and $\sigma_i^I = 1 - 0.05(i-1)$, for $i, l = 1, \ldots, 5$.

Differences for caplets with maturities from 1 to 5 years and for a wide range of strikes, are of less than 0.2%, thus being negligible if compared with typical bid-ask spreads. This result can also be interpreted by stating that already the exact price (18) can be considered a good approximation of the true model's price.

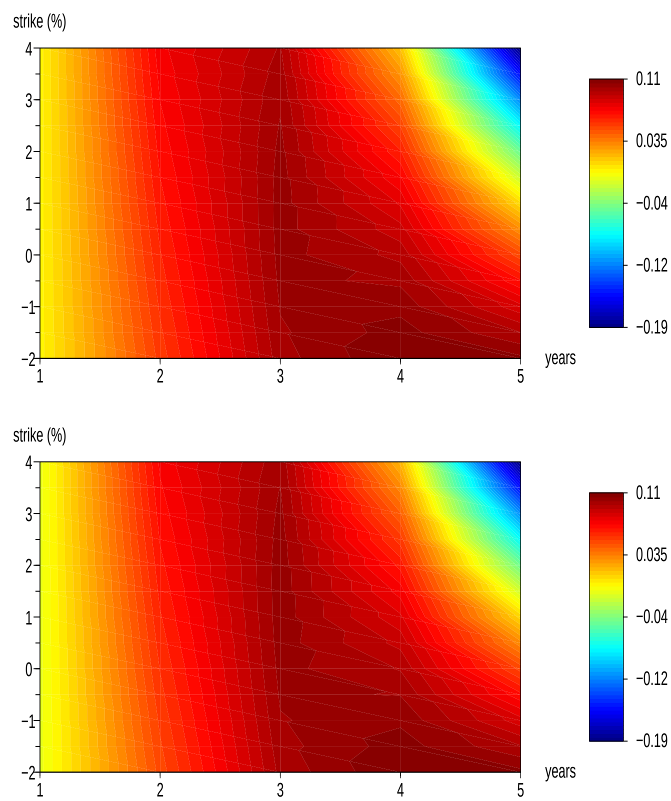

Figure 2. Percentage differences between "exact" caplets prices and freezing-based ones. The first freezing case is shown on top.

5. Example of Calibration

We now consider an example of calibration to II options, based on USD market data as of November 3, 2004, and test our stochastic volatility model under the

T_i/ Strike	1.00%	1.50%	2.00%	2.50%	3.00%	3.50%
1	178.1	134.4	95.1	62.4	37.8	21.2
2	360.4	277.4	202.9	140.3	92.0	58.0
3	539.9	419.9	312.1	221.1	150.2	99.3
4	714.5	558.9	418.9	300.4	207.4	140.0
5	886.4	696.1	524.3	378.3	263.1	179.0
6	1041.7	819.5	619.0	448.3	313.4	214.6
7	1196.8	944.7	717.1	523.0	368.8	255.1
8	1338.8	1059.5	807.5	592.3	420.9	293.8
9	1477.4	1172.6	897.6	662.4	474.6	334.6
10	1610.5	1281.9	985.4	731.6	528.3	376.0

TABLE 1. II Cap prices (in bps) for different strikes and maturities as of November 3, 2004, in the USD market.

T_i	$P(0, T_i)$	ZC rates	$\mathcal{I}_i(0)$
1y	0.97701	2.111%	194.94
2y	0.94982	2.188%	199.35
3y	0.91835	2.240%	204.03
4y	0.88433	2.278%	208.91
5y	0.84862	2.293%	213.82
6y	0.81179	2.300%	218.82
7y	0.77460	2.310%	224.00
8y	0.73785	2.320%	229.36
9y	0.70218	2.325%	234.78
10y	0.66773	2.335%	240.48

TABLE 2. USD discount factors, ZC swap rates and implied forward CPI's, on November 3, 2004.

assumption that $\rho_{i,l}^{F,I} = 0$ and $\rho_i^{F,V} = 0$ for each i, l. The quoted cap prices are reported on Table 1.

The model's price at time zero of a caplet with maturity T_i is given by formula (18) with $t = 0$, and depends on both the T_i-forward and the T_{i-1}-forward CPIs. The value at time zero of a T_i-forward CPI, where $T_i = i$ years, can be obtained from the market quote $S(T_i)$ of the corresponding zero-coupon (ZC) II swap by applying the following relation, see e.g. Mercurio (2005),

$$\mathcal{I}_i(0) = I(0)(1 + S(T_i))^i.$$

The CPI's value on November 3, 2004 was 190.91. The USD discount factors and the ZC swap rates observed on that date are reported in Table 2, together with the implied forward CPI's.

The model parameters to be calibrated are:

- The volatility parameters ϵ, α, θ, $V(0)$;
- The forward CPI's volatility coefficients σ_i^I, $i = 1, \ldots, M$;
- The correlations $\rho_{i-1,i}^I$ between consecutive forward CPI's, $i = 2, \ldots, M$;
- The correlations $\rho_i^{I,V}$ between forward CPI's and the volatility, $i = 1, \ldots, M$;

We stress that, under assumptions (12) and (13), no forward-rate parameter enters the pricing formula.

To reduce the degrees of freedom at hand, we then parameterize the CPI's correlations as

$$\rho^I_{i,i-1} = 1 - (1-\rho_0)\mathrm{e}^{-\lambda T_{i-2}}, \quad i = 2, \dots, M, \tag{26}$$

where ρ_0 is a correlation parameter and λ is a positive number. Moreover, we notice that for any $\kappa > 0$, we obtain equivalent SDE's by scaling the model parameters in (7) and (8) as follows:

$$\theta \to \kappa\theta, \;\; \epsilon \to \sqrt{\kappa}\epsilon, \;\; \sigma^I_i \to \frac{\sigma^I_i}{\sqrt{\kappa}}, \;\; V(0) \to \kappa V(0).$$

To avoid redundancies, therefore, we set $\sigma^I_1 = 1$, which corresponds to choosing $\kappa = (\sigma^I_1)^2$ as scaling factor.

Given our parameterizations and simplifications, the calibration is eventually performed on the following $2M-5$ parameters:

$$\epsilon, \alpha, \theta, V(0), \rho_0, \lambda, \sigma^I_2, \dots, \sigma^I_M, \rho^{I,V}_1, \dots, \rho^{I,V}_M.$$

The first caplet price depends on five parameters, the second on nine, and from the third on, to obtain the overall number of parameters at hand, we simply have to add two to the previous maturity's number.

Aiming to reproduce caplets prices for a wide range of strikes, we solve the calibration problem by minimizing the sum of squared percentage differences between model and market prices.

In figure 3, we report the result of our minimization by plotting the calibration errors for a reduced range of strikes and maturities. ATM prices are recovered with a very good precision, as well as the shortest maturity caplets. When we increase the caplet maturity, however, the fitting quality worsen and the improvement induced by our stochastic volatility model over the deterministic volatility LFCPIM tends to reduce. The latter effect is determined by the ergodicity of the volatility process and is in fact typical of most stochastic volatility models.

6. Conclusions

We have proposed a market model for forward inflation indices based on a stochastic volatility that evolves according to a square-root process as in Heston (1993). Closed-form formulae for option prices can be obtained in an exact fashion when the correlations between forward rates and forward CPI's and between forward rates and volatility are all zero. In the non-zero correlation case, it is still possible to derive efficient price approximations based on classical drift-freezing techniques.

We have also presented an example of calibration to II caplet prices in the USD market. We have seen that introducing stochastic volatility leads to a much better fit than in case of a deterministic one. However, postulating a unique volatility process for all forward CPI's seems too restrictive if we aim at calibrating many maturities simultaneously.

One may then consider a different stochastic volatility process for each forward CPI, introducing new volatility and correlation parameters. In this case, however, caplet prices may be difficult to derive in closed form, since the pricing of each caplet (but the first one) would involve an extra volatility process.

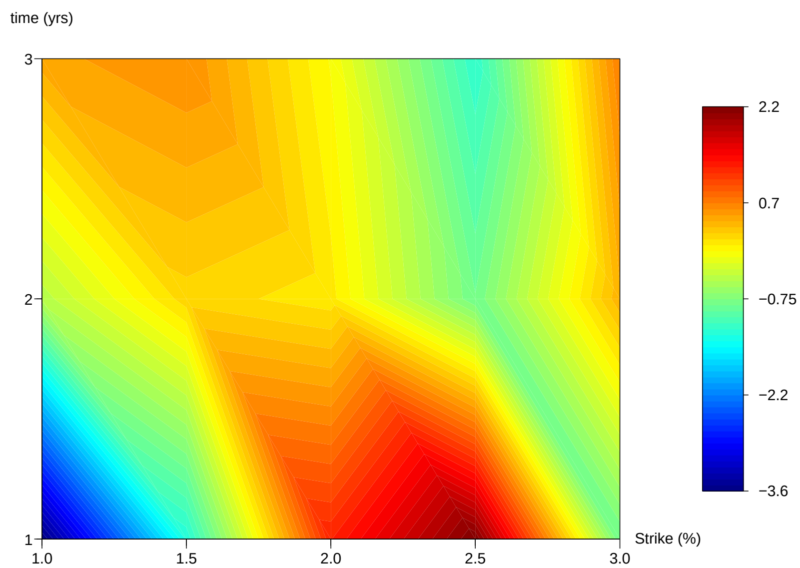

FIGURE 3. Percentage errors after calibration. Caplet maturities are equal to 1, 2 and 3 years and strikes range from 1% to 3%.

APPENDIX A. HESTON PDE

Consider the following Heston-like SDE for a stochastic process X:

$$
\begin{aligned}
dX(t) =&(\omega + \mu V(t))\, dt + \sigma \sqrt{V(t)}\, dZ(t) \\
dV(t) =&\alpha(\theta - V(t))\, dt + \epsilon \sqrt{V(t)}\, dW(t) \\
dZ(t)\, dW(t) &= \rho\, dt
\end{aligned} \tag{27}
$$

Given times t and T, $t \leq T$, we are interested in evaluating the characteristic function of the couple $(X(T), V(T))$, conditional on $\mathcal{F}_t$,

$$
\phi(X(t), V(t), t; T, u_1, u_2) = \mathbb{E}_t[\mathrm{e}^{iu_1 X(T) + iu_2 V(T)}]. \tag{28}
$$

The Feynman-Kač formula allows us to write a PDE satisfied by ϕ:

$$
\begin{cases}
\dfrac{\partial \phi}{\partial t} + [\alpha\theta - \alpha v]\dfrac{\partial \phi}{\partial v} + [\omega + v\mu]\dfrac{\partial \phi}{\partial x} + \dfrac{1}{2}\dfrac{\partial^2 \phi}{\partial x^2} v\sigma^2 + \dfrac{1}{2}\epsilon^2 v \dfrac{\partial^2 \phi}{\partial v^2} + \sigma\epsilon\rho v \dfrac{\partial^2 \phi}{\partial v \partial x} = 0 \\
\phi(x, v, t; T, u_1, u_2) = \mathrm{e}^{iu_1 x + iu_2 v}
\end{cases} \tag{29}
$$

Following Heston (1993), we guess a solution like

$$
\begin{aligned}
\phi(x, v, t; T, u_1, u_2) &= \exp[A(T - t, u_1, u_2) + B(T - t, u_1, u_2)v + iu_1 x] \\
A(0, u_1, u_2) &= 0 \\
B(0, u_1, u_2) &= iu_2,
\end{aligned} \tag{30}
$$

which is suggested by the linearity in v of the PDE's coefficients. The PDE in Heston (1993) is recovered by setting $u_2 = 0$.

Let us set $\tau := T - t$. Once we plug our *ansatz* solution into (29), we have (for simplicity, we drop dependence on u_1, u_2)

$$\begin{aligned}\frac{\partial A(\tau)}{\partial \tau} &= B(\tau)\alpha\theta + iu_1\omega \\ \frac{\partial B(\tau)}{\partial \tau} &= \frac{\epsilon^2}{2}B^2(\tau) + B(\tau)(iu_1\sigma\epsilon\rho - \alpha) + iu_1\mu - \frac{1}{2}\sigma^2 u_1^2.\end{aligned} \tag{31}$$

The second equation is in the form

$$\frac{\partial B(\tau)}{\partial \tau} = aB^2(\tau) + bB(\tau) + c, \qquad a, b, c \in \mathbb{C} \tag{32}$$

and can be solved, for instance,[6] by setting $B_\pm = \frac{1}{2a}\left[-b \pm \sqrt{b^2 - 4ac}\right] := \frac{1}{2a}(-b \pm \gamma)$, $\gamma = \sqrt{b^2 - 4ac}$ and by rewriting (31) as

$$\left(\frac{1}{B - B_+} - \frac{1}{B - B_-}\right)\partial B = \gamma\partial\tau. \tag{33}$$

After some algebra, we have, for each $\tau > 0$,

$$\begin{aligned}B(\tau) &= \frac{(\gamma - b)}{2a}\,\frac{1 - \frac{b+\gamma}{b-\gamma}\cdot\frac{2aB(0)+b-\gamma}{2aB(0)+b+\gamma}\mathrm{e}^{\gamma\tau}}{1 - \frac{2aB(0)+b-\gamma}{2aB(0)+b+\gamma}\mathrm{e}^{\gamma\tau}} \\ &= B(0) + \frac{\gamma - b - 2aB(0)}{2a}\left[\frac{1 - \mathrm{e}^{\gamma\tau}}{1 - \frac{2aB(0)+b-\gamma}{2aB(0)+b+\gamma}\mathrm{e}^{\gamma\tau}}\right]\end{aligned} \tag{34}$$

which coincides with the corresponding Heston's formula. Function A is then obtained by integration

$$A(\tau) - A(0) = iu_1\omega\tau + \frac{\alpha\theta(\gamma - \beta)}{2a\gamma}\int_1^{\mathrm{e}^{\gamma\tau}}\left[\frac{1}{y} + \frac{f - d}{1 - fy}\right]dy$$

with $f = (2aB(0) + b - \gamma)/(2aB(0) + b + \gamma)$ and $d = f(b + \gamma)/(b - \gamma)$, so that, finally,

$$A(\tau) = A(0) + \left[iu_1\omega + \frac{\alpha\theta(\gamma - b)}{2a}\right]\tau - \frac{\alpha\theta}{a}\ln\left[\frac{1 - \frac{2aB(0)+b-\gamma}{2aB(0)+b+\gamma}\mathrm{e}^{\gamma\tau}}{1 - \frac{2aB(0)+b-\gamma}{2aB(0)+b+\gamma}}\right] \tag{35}$$

with

$$\begin{aligned}a &= \epsilon^2/2 \\ b &= iu_1\sigma\epsilon\rho - \alpha \\ c &= iu_1\mu - \sigma^2 u_1^2/2 \\ A(0) &= 0 \\ B(0) &= iu_2.\end{aligned}$$

[6]See Wu and Zhang (2002) [9] for an equivalent derivation.

Appendix B. Floorlet Pricing

Let us consider a T_j-maturity floorlet, whose time-t value is given by

$$\mathrm{Flrt}_j(t,K) = P(t,T_j)\int_{-\infty}^{+\infty} (\mathrm{e}^k - \mathrm{e}^s)^+ q_t^j(s)ds = P(t,T_j)\int_{-\infty}^{k} (\mathrm{e}^k - \mathrm{e}^s) q_t^j(s)ds$$

where $k = \ln(K+1)$ and q_t^j is defined in Section 4.

To price a call option with an FT approach, Carr and Madan (1998) tackled the non-integrability issue of the call price for $k \to -\infty$, by introducing a regularizing factor leading to a modified caplet price that tends to zero whenever $k \to -\infty$. Here, instead, we have that the floorlet price, as function of k, is not square integrable on the positive semi-axis, when $k \to +\infty$. We thus choose $\tilde\eta > 1$ and introduce an exponential dumping factor by setting

$$\mathrm{Flrt}_j(t,K) = P(t,T_j)\mathrm{e}^{\tilde\eta k} H(k) := P(t,T_j)\mathrm{e}^{\tilde\eta k}\int_{-\infty}^{k} \mathrm{e}^{-\tilde\eta k}(\mathrm{e}^k - \mathrm{e}^s) q_t^j(s)ds. \tag{36}$$

It is now easy to compute the FT $\hat H(u)$ of $H(k)$ in terms of the characteristic function ϕ_t^j as

$$\begin{aligned}\hat H(u) &= \int_{-\infty}^{+\infty} dk\, \mathrm{e}^{iku} H(k) = \int_{-\infty}^{+\infty} dk \int_{-\infty}^{k} ds\, \mathrm{e}^{(iu-\tilde\eta)k}(\mathrm{e}^k - \mathrm{e}^s) q_t^j(s) \\ &= \int_{-\infty}^{+\infty} ds\, q_t^j(s) \int_{s}^{+\infty} dk\, \mathrm{e}^{(iu-\tilde\eta)k}(\mathrm{e}^k - \mathrm{e}^s) \\ &= \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{i[u+i(\tilde\eta-1)]s}}{(iu-\tilde\eta+1)(iu-\tilde\eta)} q_t^j(s)\, ds = \frac{\phi_t^j(u+i(\tilde\eta-1))}{(iu-\tilde\eta+1)(iu-\tilde\eta)}.\end{aligned} \tag{37}$$

The floorlet price is then calculated by means of the inverse FT:

$$\mathrm{Flrt}_j(t,K) = P(t,T_j)\frac{\mathrm{e}^{\tilde\eta k}}{\pi}\mathrm{Re}\int_0^{+\infty} \mathrm{e}^{-iuk}\frac{\phi_t^j(u+i(\tilde\eta-1))}{(iu-\tilde\eta+1)(iu-\tilde\eta)}\, du \tag{38}$$

We finally remark that, also for II options, it is possible to derive an explicit put-call parity. In fact, since $(x-y)^+ - (y-x)^+ = x - y$, we have that,

$$\begin{aligned}\mathrm{Cplt}_j(t,K) - \mathrm{Flrt}_j(t,K) &= P(t,T_j)\mathbb{E}_t^j\left\{\frac{\mathcal{I}_j(T_j)}{\mathcal{I}_{j-1}(T_{j-1})} - (K+1)\right\} \\ &= P(t,T_j)\left\{\mathbb{E}_t^j\left[\frac{\mathcal{I}_j(T_{j-1})}{\mathcal{I}_{j-1}(T_{j-1})}\right] - (K+1)\right\} \\ &= P(t,T_j)\left\{\mathbb{E}_t^j\left[\mathrm{e}^{X_j(T_{j-1})}\right] - (K+1)\right\},\end{aligned} \tag{39}$$

where the second equality holds since $\mathcal{I}_j$ is a martingale under $\mathbb{Q}^j$. The residual expectation in (39) is the characteristic function of $X_j(T_{j-1})$ evaluated at point $-i$, and can be easily computed as explained in Appendix A and in Section 4.1.

References

[1] Andersen, L., Torben, G., and Jesper, L. (1997). Estimating continuous-time stochastic volatility models of the short-term interest rate, *Journal of Econometrics*, Vol 77(2), pp. 343-377

[2] Belgrade, N., Benhamou, E., and Koehler E. (2004). A Market Model for Inflation , *SSRN Working Paper*, available online at http://ssrn.com/abstract=576081

[3] Carr, P., and Madan, D. (1998). Option Valuation Using the Fast Fourier Transform, *Working Paper, Morgan Stanley and University of Maryland*
[4] Geman, H., El Karoui, N., and Rochet, J.-C. (1995) Changes of Numeraire, Changes of Probability Measure and Option Pricing. *Journal of Applied Probability* 32, 443-458.
[5] Heston, S.L. (1993). A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *The Review of Financial Studies, vol.6, n.2, pp.327-343*
[6] Jarrow, R., and Yildirim Y. (2003). Pricing Treasury Inflation Protected Securities and Related Derivatives using an HJM Model. *Journal of Financial and Quantitative Analysis* 38(2), 409-430.
[7] Kazziha, S. (1999). Interest Rate Models, Inflation-based Derivatives, Trigger Notes And Cross-Currency Swaptions. PhD Thesis, Imperial College of Science, Technology and Medicine. London.
[8] Mercurio, F. (2005) Pricing Inflation-Indexed Derivatives, *Quantitative Finance*, 5(3). Forthcoming.
[9] Wu L. and Zhang F. (2002) Libor Market Model: from Deterministic to Stochastic Volatility, *Working Paper Claremont Graduated University and Hong Kong University of Science and Technology*

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