ALTERNATIVE ASSET-PRICE DYNAMICS AND VOLATILITY SMILES

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STYLIZED FACTS MOTIVATING the PAPER

- Traders use the Black-Scholes formula to price plain-vanilla options.
- Options are priced through their implied volatility. This is the $\sigma$ parameter to plug into the Black-Scholes formula to match the corresponding market price:

$$S_0e^{-qT}\Phi\left(\frac{\ln \frac{S_0}{K} + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\ln \frac{S_0}{K} + (r - q - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) = C(K, T)$$

- Implied volatilities vary with strike and maturity. They are skew-shaped (low-strikes vols are higher than high-strikes vols) or smile-shaped (the volatility is minimum around the underlying forward price).
STYLIZED FACTS MOTIVATING the PAPER (cont’d)

Consequences:

• The Black-Scholes model cannot consistently price all options traded in a market (the risk-neutral distribution is not lognormal).

• Need for an alternative asset price model to price exotics or non quoted plain-vanilla options.

• The model should:
  – Feature explicit asset-price dynamics with a known marginal distribution.
  – Imply analytical formulas for European options.
  – Imply a good fitting of market data (reasonable number of parameters).
  – Be stable enough.
MAJOR REFERENCES

We tackle the issue of pricing general volatility structures by assuming a suitable local volatility model.

References:


The RELATED LITERATURE

FIRST APPROACH: Alternative Explicit Dynamics
• It immediately leads to volatility smiles or skews.
• Examples: The general CEV process (Cox (1975) and Cox and Ross (1976)). A general class of processes is due to Carr, Tari and Zariphopoulou (1999).

SECOND APPROACH: Continuum of Traded Strikes
• It goes back to Breeden and Litzenberger (1978).

THIRD APPROACH: Lattice Approach
• Based on finding the risk-neutral probabilities in a tree that best fit market prices due to some smoothness criterion.
The RELATED LITERATURE (cont’d)

FOURTH APPROACH: *Incomplete Market*


FIFTH APPROACH: *Market Model*

- Analogous to the Market Model for interest rates
PRICING the SMILE for RISK MANAGEMENT PURPOSES

Assume we hold a one-year maturity call with strike 90 and underlying stock price 100. If \( r = 0.05 \) and \( \sigma = 0.3 \), the call price is:

\[
BS(S_0 = 100, K = 90, \sigma = 0.3) = 19.697
\]

Now assume that the stock price drops to 90. The call price becomes:

\[
BS(S_0 = 90, K = 90, \sigma = 0.3) = 12.808;
\]

However, if we take into account the volatility smile:

\[
BS(S_0 = 90, K = 90, \sigma = 0.2) = 9.406;
\]
An ANALYTICALLY TRACTABLE CLASS of MODELS

We propose a class of analytically tractable models for an asset-price dynamics that are flexible enough to reproduce a large variety of market volatility structures.

The asset underlies a given option market (needs not be tradable). We can think of an exchange rate, a stock index, and even a forward LIBOR rate.

We assume that:

- The $T$-forward risk-adjusted measure $Q^T$ exists.
- The dynamics of the asset price $S$ under $Q^T$ is
  \[ dS_t = \mu S_t dt + \sigma(t, S_t) S_t dW_t, \quad S_0 > 0, \]
  where $\mu$ is a constant and $\sigma$ is well behaved.
- The marginal density of $S$ under $Q^T$ is equal to the weighted average of the known densities of some given diffusion processes.
Let us then consider $N$ diffusion processes with dynamics given by

$$dS^i_t = \mu S^i_t dt + v_i(t, S^i_t) dW_t, \quad S^i_0 = S_0,$$

where $v_i(t, y)$’s are real functions satisfying regularity conditions to ensure existence and uniqueness of the solution to the SDE.

For each $t$, we denote by $p^i_t(\cdot)$ the density function of $S^i_t$ ($p^i_0(y)$ is the Dirac-$\delta$ function centered in $S^i_0$).

**Problem.** Derive the local volatility $\sigma(t, S_t)$ such that the $Q^T$-density of $S_t$ satisfies

$$p_t(y) := \sum_{i=1}^N \lambda_i p^i_t(y),$$

where $\lambda_i$’s are (strictly) positive constants such that $\sum_{i=1}^N \lambda_i = 1$. 
An ANALYTICALLY TRACTABLE CLASS of MODELS: the PROBLEM SOLUTION

N.B. $p_t(\cdot)$ is a proper $Q^T$-density function:

$$
\int_0^{+\infty} yp_t(y) dy = \sum_{i=1}^{N} \lambda_i \int_0^{+\infty} yp_t^i(y) dy = \sum_{i=1}^{N} \lambda_i S_0 e^{\mu t} = S_0 e^{\mu t}
$$

Solution. Apply the Fokker-Planck equation (FPE)

$$
\frac{\partial}{\partial t} p_t(y) = -\frac{\partial}{\partial y} (\mu y p_t(y)) + \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( \sigma^2(t, y) y^2 p_t(y) \right),
$$

to back out $\sigma$, given that the FPE holds for each basic process as well.

Applying the definition of $p_t(y)$, the linearity of the derivative operator and the above FPEs, we get:

$$
dS_t = \mu S_t dt + \sqrt{\frac{\sum_{i=1}^{N} \lambda_i \sigma_i^2(t, S_t)p_t^i(S_t)}{\sum_{i=1}^{N} \lambda_i S_t^2 p_t^i(S_t)}} S_t dW_t
$$

N.B. This SDE, however, only defines some candidate dynamics leading to the marginal density $p_t(\cdot)$. 
An ANALYTICALLY TRACTABLE CLASS of MODELS: OPTION PRICING

Let us give for granted that the previous SDE has a unique strong solution and consider a European option with maturity $T$, strike $K$ and written on the asset.

The option value at time $t = 0$ is ($\omega = \pm 1$):

$$\mathcal{O} = P(0, T)E^T \{[\omega(S_T - K)]^+\}$$

$$= P(0, T) \int_0^{+\infty} [\omega(y - K)]^+ \sum_{i=1}^N \lambda_i p_T^i(y) dy$$

$$= \sum_{i=1}^N \lambda_i P(0, T) \int_0^{+\infty} [\omega(y - K)]^+ p_T^i(y) dy = \sum_{i=1}^N \lambda_i \mathcal{O}_i$$

**Remark [Greeks].** The same convex combination applies also to all option Greeks.

**Remark [Why a mixture of densities?]** i) if $p^i$'s are analytically tractable, we immediately have closed-form formulas for European options; ii) the number of model parameters is virtually unlimited.
The LOGNORMAL-MIXTURE CASE

We now assume that, for each $i$,

$$v_i(t, y) = \sigma_i(t)y,$$

where $\sigma_i$’s are deterministic, continuous and bounded from below by positive constants.

We also assume there exists an $\varepsilon > 0$ such that $\sigma_i(t) = \sigma_0 > 0$, for each $t$ in $[0, \varepsilon]$ and $i = 1, \ldots, N$.

Remark [Why a mixture of lognormals?]

• It is analytically tractable and obviously linked to the Black-Scholes model.

• The log-returns $\ln(S_t/S_0)$, $t > 0$, are more leptokurtic than in the Gaussian case.

The LOGNORMAL-MIXTURE CASE: the ASSET-PRICE DYNAMICS

Proposition. If we set \( V_i(t) := \sqrt{\int_0^t \sigma_i^2(u) \, du} \) and

\[
\nu(t, y) = \sqrt{\sum_{i=1}^N \lambda_i \sigma_i^2(t) \frac{1}{V_i(t)} \exp \left\{ -\frac{1}{2V_i^2(t)} \left[ \ln \frac{y}{S_0} - \mu t + \frac{1}{2} V_i^2(t) \right]^2 \right\}} \sum_{i=1}^N \lambda_i \frac{1}{V_i(t)} \exp \left\{ -\frac{1}{2V_i^2(t)} \left[ \ln \frac{y}{S_0} - \mu t + \frac{1}{2} V_i^2(t) \right]^2 \right\},
\]

for \((t, y) > (0, 0)\) and \(\nu(0, S_0) := \sigma_0\) for, the SDE

\[
dS_t = \mu S_t dt + \nu(t, S_t) S_t dW_t,
\]

has a unique strong solution whose marginal density is

\[
p_t(y) = \sum_{i=1}^N \lambda_i \frac{1}{y V_i(t) \sqrt{2\pi}} \exp \left\{ -\frac{1}{2V_i^2(t)} \left[ \ln \frac{y}{S_0} - \mu t + \frac{1}{2} V_i^2(t) \right]^2 \right\}.
\]

N.B. We notice that for \((t, y) > (0, 0)\)

\[
\nu^2(t, y) = \sum_{i=1}^N \Lambda_i(t, y) \sigma_i^2(t),
\]

where, \(\Lambda_i(t, y) \geq 0\) and \(\sum_{i=1}^N \Lambda_i(t, y) = 1\). Therefore:

\[
0 < \tilde{\sigma} \leq \nu(t, y) \leq \hat{\sigma} < +\infty \text{ for each } t, y > 0.
\]
The LOGNORMAL-MIXTURE CASE: the ASSET-PRICE DYNAMICS (cont’d)

Proof. The proof is based on the Theorem 12.1 in Section V.12 of Rogers and Williams (1996).

We write \( S_t = \exp(Z_t) \), where
\[
dZ_t = \left[ \mu - \frac{1}{2} \sigma^2 (t, e^{Z_t}) \right] dt + \sigma (t, e^{Z_t}) dW_t,
\]
The coefficients of this SDE are bounded, and hence satisfy the usual linear-growth condition.

Setting \( u(t, z) := \sigma(t, e^{z}) \), we have that \( \frac{\partial u^2}{\partial z}(t, z) \) is well defined and continuous for \((t, z) \in (0, M] \times \mathbb{R}, M > 0 \) (continuity of \( \sigma_i \) and \( V_i \)), and (\( u \) is constant for \( t \in [0, \epsilon] \))
\[
\lim_{t \to 0} \frac{\partial u^2}{\partial z}(t, z) = 0.
\]
The derivative \( \frac{\partial u^2}{\partial z}(t, z) \) is thus bounded on each compact set \([0, M] \times [-M, M]\), and so is \( \frac{\partial u}{\partial z}(t, z) = \frac{1}{2u(t, z)} \frac{\partial u^2}{\partial z}(t, z) \) since \( \sigma \) is bounded from below. Hence, \( u \) is locally Lipschitz.
Theorem 12.1 in Section V.12 of Rogers and Williams (1996)

Suppose that the coefficients $\sigma$ and $b$ in the SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

are such that for each $N$ there is some $K_N$ such that:

$$|\sigma(s, x) - \sigma(s, y)| \leq K_N|x - y|$$
$$|b(s, x) - b(s, y)| \leq K_N|x - y|$$

whenever $\max(|x|, |y|) \leq N$ and $0 \leq s \leq N$. Suppose also that for each constant $T > 0$, there is some $C_T$ such that, for $0 \leq s \leq T$,

$$|\sigma(s, x)| + |b(s, x)| \leq C_T(1 + |x|)$$

then the above SDE is an exact SDE.
The LOGNORMAL-MIXTURE CASE: OPTION PRICING

**Proposition.** The time-0 price of a European option with maturity $T$, strike $K$ and written on the asset is

$$
\mathcal{O} = \omega P(0, T) \sum_{i=1}^{N} \lambda_i \left[ S_0 e^{\mu T} \Phi \left( \omega \frac{\ln \frac{S_0}{K} + (\mu + \frac{1}{2} \eta_i^2) T}{\eta_i \sqrt{T}} \right) 
- K \Phi \left( \omega \frac{\ln \frac{S_0}{K} + (\mu - \frac{1}{2} \eta_i^2) T}{\eta_i \sqrt{T}} \right) \right],
$$

where $\omega = 1(-1)$ for a call (put), and $\eta_i := \frac{V_i(T)}{\sqrt{T}}$.

For each $T$, the implied volatility is smile-shaped:

![Graph](image)

Figure 1: $\mu = .035, T = 1, (V_1(1), V_2(1), V_3(1)) = (.5, .1, .2), (\lambda_1, \lambda_2, \lambda_3) = (.2, .3, .5), S_0 = 100.$
**The LOGNORMAL-MIXTURE CASE: the IMPLIED VOLATILITY**

**Definition.** Defining the moneyness $m$ by

$$m := \ln \frac{S_0}{K} + \mu T,$$

the Black-Scholes volatility $\hat{\sigma}(m)$ that, for a given $T$, is implied by the above option price is implicitly given by

$$P(0,T)S_0e^\mu T \left[ \Phi\left( \frac{m + \frac{1}{2}\hat{\sigma}(m)^2 T}{\hat{\sigma}(m)\sqrt{T}} \right) - e^{-m} \Phi\left( \frac{m - \frac{1}{2}\hat{\sigma}(m)^2 T}{\hat{\sigma}(m)\sqrt{T}} \right) \right]$$

$$= P(0,T)S_0e^\mu T \sum_{i=1}^{N} \lambda_i \left[ \Phi\left( \frac{m + \frac{1}{2}\eta_i^2 T}{\eta_i \sqrt{T}} \right) - e^{-m} \Phi\left( \frac{m - \frac{1}{2}\eta_i^2 T}{\eta_i \sqrt{T}} \right) \right]$$

**Proposition.** The Black-Scholes volatility implied by the above option price is (neglecting $o(m^2)$ terms):

$$\hat{\sigma}(m) = \hat{\sigma}(0) + \frac{1}{2\hat{\sigma}(0)T} \sum_{i=1}^{N} \lambda_i \left[ \frac{1}{\eta_i} e^{\frac{1}{8}(\hat{\sigma}(0)^2 - \eta_i^2)T} - 1 \right] m^2$$

where the ATM-forward implied volatility $\hat{\sigma}(0)$ is

$$\hat{\sigma}(0) = \frac{2}{\sqrt{T}} \Phi^{-1} \left( \sum_{i=1}^{N} \lambda_i \Phi\left( \frac{1}{2\eta_i \sqrt{T}} \right) \right).$$
SHIFTING the DISTRIBUTION

Let us define a new asset-price process $A$ by:

$$A_t = A_0 \alpha e^{\mu t} + S_t,$$

where $\alpha$ is a real constant.

By Ito’s formula, we immediately obtain:

$$dA_t = \mu A_t dt + \nu(t, A_t - A_0 \alpha e^{\mu t})(A_t - A_0 \alpha e^{\mu t})dW_t.$$

Some possible densities

![Figure 2: The density function $p(x)$ of $A_T$ for the different values of $\alpha \in \{-0.4, -0.2, 0, 0.2\}$, where we set $A_0 = 100$, $\mu = 0.05$, $T = 0.5$, $N = 3$, $(\eta_1(T), \eta_2(T), \eta_3(T)) = (0.25, 0.09, 0.04)$ and $(\lambda_1, \lambda_2, \lambda_3) = (0.8, 0.1, 0.1)$.

]
SHIFTING the DISTRIBUTION: OPTION PRICING

Proposition. The time-0 price of a European option with strike $K$, maturity $T$ and written on the asset is

$$
\mathcal{O} = \omega P(0, T) \sum_{i=1}^{N} \lambda_i \left[ A_0 e^{\mu T} \Phi\left( \omega \frac{\ln A_0}{K} + \left( \mu + \frac{1}{2} \eta_i^2 \right) T \right) \right.

- \mathcal{K} \Phi\left( \omega \frac{\ln A_0}{K} + \left( \mu - \frac{1}{2} \eta_i^2 \right) T \right) \right],

$$

where $\mathcal{K} = K - A_0 \alpha e^{\mu T}$, $A_0 = A_0(1 - \alpha)$. Moreover:

$$
\hat{\sigma}(m) = \hat{\sigma}(0) + \alpha \left[ \frac{\sum_{i=1}^{N} \lambda_i \Phi\left( -\frac{1}{2} \eta_i \sqrt{T} \right) - \frac{1}{2}}{\sqrt{T} e^{-\frac{1}{8} \hat{\sigma}(0)^2 T} m} \right.

+ \frac{1}{2} \left[ \frac{1}{T(1 - \alpha)} \sum_{i=1}^{N} \frac{\lambda_i}{\eta_i} e^{\frac{1}{8} \hat{\sigma}(0)^2 - \eta_i^2 T} - \frac{1}{\hat{\sigma}(0)} T \right.

+ \frac{\alpha^2}{4} \hat{\sigma}(0) T \left( \frac{\sum_{i=1}^{N} \lambda_i \Phi\left( -\frac{1}{2} \eta_i \sqrt{T} \right) - \frac{1}{2}}{\sqrt{T} e^{-\frac{1}{8} \hat{\sigma}(0)^2 T}} \right)^2 m^2 + o(m^2)

$$

where the ATM-forward implied volatility $\hat{\sigma}(0)$ is now

$$
\hat{\sigma}(0) = \frac{2}{\sqrt{T}} \Phi^{-1}\left( (1 - \alpha) \sum_{i=1}^{N} \lambda_i \Phi\left( \frac{1}{2} \eta_i \sqrt{T} \right) + \frac{\alpha}{2} \right).
$$
SHIFTING the DISTRIBUTION: the IMPACT of $\alpha$

Decreasing $\alpha$, the variance of the asset-price at each time increases while maintaining the correct expectation:

$$E(A_t) = A_0 e^{\mu t}$$

$$\text{Var}(A_t) = A_0^2 (1 - \alpha)^2 e^{2\mu t} \left( \sum_{i=1}^{N} \lambda_i e^{V_i^2(t)} - 1 \right).$$

- $\alpha$ concurs to determine the implied-volatility level.
- $\alpha$ moves the strike where the volatility is minimum.

Figure 3: $A_0 = 100$, $\mu = 0.05$, $N = 2$, $T = 2$, $K \in [80, 120]$. Left: implied volatility curve for $(\eta_1(T), \eta_2(T)) = (0.35, 0.1)$, $(\lambda_1, \lambda_2) = (0.6, 0.4)$, $\alpha \in \{0, -0.2, -0.4\}$; Right: implied volatility curve for $(\lambda_1, \lambda_2) = (0.6, 0.4)$ in the three cases: a) $(\eta_1(T), \eta_2(T)) = (0.35, 0.1)$, $\alpha = 0$; b)$(\eta_1(T), \eta_2(T)) = (0.110, 0.355)$, $\alpha = -0.2$; c)$(\eta_1(T), \eta_2(T)) = (0.098, 0.298)$, $\alpha = -0.4$;
APPLYING the MODEL in PRACTICE

The asset is a stock/index.

- Assume constant interest rates (all equal to $r > 0$), and set $\mu = r - q$ ($q$ is the dividend yield).
- Pronounced skews can be produced (but not highly steep curves for very short maturities).

The asset is an exchange rate.

- Assume constant interest rates, and set $\mu = r - r_f$ ($r_f$ is the foreign risk-free rate).
- Exchange rate volatilities are typically smile-shaped.

The asset is a forward LIBOR rate.

- The forward LIBOR rate at time $t$ for $[S, T]$ is
  \[ F(t, S, T) = \frac{1}{\tau(S, T)} \left[ \frac{P(t, S)}{P(t, T)} - 1 \right], \]
  where $\tau(S, T)$ is the year fraction from $S$ to $T$.
- Since $F(\cdot, S, T)$ is a martingale under $Q^T$, set $\mu = 0$. 
The CALIBRATION to MARKET DATA

General comments

- The (virtually unlimited) number of parameters can render the calibration to market data very accurate.
- When minimizing the “distance” between model and market prices, the search for a global minimum can be cumbersome and inevitably slow.
- The use of a local-search algorithm speeds up the calibration process (in practice, it is advisable to combine the two searches).

Getting ready for a calibration

- Assume we have $M$ option maturities $T_1 < \ldots < T_M$.
- Set $v_{i,j} := \eta_i(T_j)$ for $i = 1, \ldots, N$ and $j = 1, \ldots, M$.
- Impose the constraints $v_{i,j+1} > v_{i,j} \sqrt{T_j/T_{j+1}}$, $\forall i, j$.
- More constraints: $K > A_0 \alpha e^{\mu T}$, for each “traded” $K$.
- We may assume that $v_{i,j} = \bar{v}_i > 0$.
- It is enough to set $N = 2, 3$ in most applications.
FIRST EXAMPLE of CALIBRATION to REAL MARKET DATA

Data: two-year Euro caplet volatilities as of November 14th, 2000 (LIBOR resetting at 1.5 years).

We set: \( N = 2, v_i := \eta_i(1.5), i = 1, 2, \lambda_2 = 1 - \lambda_1. \)

We minimize the squared percentage difference between model and market (mid) prices. We get: \( \lambda_1 = 0.241, \lambda_2 = 0.759, v_1 = 0.125, v_2 = 0.194, \alpha = 0.147. \)
SECOND EXAMPLE of CALIBRATION to REAL MARKET DATA

Data: Italian MIB30 equity index on March 29, 2000, at 3:21pm (most liquid puts with the shortest maturity).

We set $N = 3$, $v_i := \eta_i(T)$ ($i = 1, 2, 3$), $\lambda_3 = 1 - \lambda_1 - \lambda_2$. We minimize the squared percentage difference between model and market mid prices. We get: $\lambda_1 = 0.201$, $\lambda_2 = 0.757$, $v_1 = 0.019$, $v_2 = 0.095$, $v_3 = 0.229$, $\alpha = -1.852$. 
THIRD EXAMPLE of CALIBRATION to REAL MARKET DATA


We set $N = 2$, $v_i := \eta_i(0.167)$ ($i = 1, 2$), $\lambda_2 = 1 - \lambda_1$. We minimize the squared percentage difference between model and market mid prices. We get: $\lambda_1 = 0.451$, $v_1 = 0.129$, $v_2 = 0.114$, $\alpha = 0.076$. 

![Graph showing market volatilities and calibrated volatilities]
APPLICATION to the FORWARD-LIBOR MARKET MODEL

Proposition. The dynamics of \( F_k := F(\cdot; T_{k-1}, T_k) \) under the forward measure \( Q^i \) in the three cases \( i < k \), \( i = k \) and \( i > k \) are, respectively,

\[ i < k, \ t \leq T_i : \]
\[ dF_k(t) = \nu_k(t, F_k(t)) F_k(t) \sum_{j=i+1}^{k} \frac{\rho_{k,j} \tau_j \nu_j(t, F_j(t)) F_j(t)}{1 + \tau_j F_j(t)} \, dt \]
\[ + \nu_k(t, F_k(t)) F_k(t) \, dZ_k(t), \]

\[ i = k, \ t \leq T_{k-1} : \]
\[ dF_k(t) = \nu_k(t, F_k(t)) F_k(t) \, dZ_k(t), \]

\[ i > k, \ t \leq T_{k-1} : \]
\[ dF_k(t) = -\nu_k(t, F_k(t)) F_k(t) \sum_{j=k+1}^{i} \frac{\rho_{k,j} \tau_j \nu_j(t, F_j(t)) F_j(t)}{1 + \tau_j F_j(t)} \, dt \]
\[ + \nu_k(t, F_k(t)) F_k(t) \, dZ_k(t), \]
where \( Z = Z^i \) is a Brownian motion under \( Q^i \). All the above equations admit a unique strong solution.
FURTHER EXTENSIONS: a LOGNORMAL MIXTURE with DIFFERENT MEANS

Let us consider the instrumental processes

\[ dS_t^i = \mu_i(t)S_t^i dt + \sigma_i(t)S_t^i dW_t, \quad S_0^i = S_0, \]

and look for a diffusion coefficient \( \psi(\cdot, \cdot) \) such that

\[ dS_t = \mu S_t dt + \psi(t, S_t) S_t dW_t \]

has a solution with marginal density \( p_t(y) = \sum_{i=1}^{m} \lambda_i p^i_t(y) \).

Denote by \( \nu(t, S_t) \) the solution of the analogous problem when the basic processes have the same drift \( \mu \):

\[ \nu(t, y)^2 = \frac{\sum_{i=1}^{m} \lambda_i \sigma_i(t)^2 p^i_t(y)}{\sum_{i=1}^{m} \lambda_i p^i_t(y)} \]

It is then possible to show that

\[ \psi(t, y)^2 := \nu(t, y)^2 + \frac{2 \sum_{i=1}^{m} \lambda_i (\mu_i(t) - \mu) \int_{y}^{\infty} xp^i_t(x) dx}{y^2 \sum_{i=1}^{m} \lambda_i p^i_t(y)} \]

It is also possible to prove that \( \psi(\cdot, \cdot) \) has linear growth and does not explode in finite time.
FURTHER EXTENSIONS: the CASE of HYPERBOLIC-SINE BASIC PROCESSES

Consider now the instrumental processes, \( i = 1, \ldots, N \),

\[
S_i(t) = \beta_i(t) \sinh \left[ \int_0^t \alpha_i(u) dW_u - L_i \right], \quad S_i(0) = S_0
\]

where \( \alpha_i \)'s are positive functions, \( L_i \)'s are negative constants and

\[
\beta_i(t) = \frac{S_0 e^{\mu t - \frac{1}{2} \int_0^t \alpha_i^2(u) du}}{\sinh(-L_i)}.
\]

N.B. \( S_i \) is an increasing function of a time-changed Brownian motion.

The SDE followed by each \( S_i \) is given by

\[
dS_i(t) = \mu S_i(t) dt + \alpha_i(t) \sqrt{\beta_i^2(t) + S_i^2(t)} dW_t.
\]

Setting \( A_i(t) := \sqrt{\int_0^t \alpha_i^2(u) du} \), the time-\( t \) marginal density of \( S_i \) is

\[
p^i_t(y) = \frac{\exp \left\{ -\frac{1}{2} A_i^2(t) \left[ L_i + \sinh^{-1} \left( \frac{y}{\beta_i(t)} \right) \right]^2 \right\}}{A_i(t) \sqrt{2\pi} \sqrt{\beta_i^2(t) + y^2}}.
\]
The CASE of HYPERBOLIC-SINE BASIC PROCESSES (cont’d)

A straightforward integration leads to the call price:

\[
C(T, K) = P(0, T) \left[ \frac{S_0 e^{\mu T}}{2 \sinh(-L_i)} \left( e^{-L_i \Phi(\bar{y}_i(T)) + A_i(T)} - e^{L_i \Phi(\bar{y}_i(T) - A_i(T))} \right) - K \Phi(\bar{y}_i(T)) \right],
\]

where we set

\[
\bar{y}_i(T) := -\frac{L_i}{A_i(T)} - \frac{1}{A_i(T)} \sinh^{-1} \left( \frac{K}{\beta_i(T)} \right).
\]

This price leads to steeply decreasing implied volatilities:

Figure 4: We set, \( T = 1, A_1(1) = 0.01, L_1 = -0.05, \mu = 0 \) and \( S_0 = 0.055 \).
The CASE of HYPERBOLIC-SINE BASIC PROCESSES: OPTION PRICING

The previous general results on densities-mixture dynamics immediately yield the following SDE:

\[ dS(t) = \mu S(t) dt + \chi(t, S(t)) \, dW_t \]

\[ \chi(t, y) = \sqrt{\frac{\sum_{i=1}^{N} \lambda_i \frac{a_i^2(t) \sqrt{\beta_i(t)^2 + y^2}}{A_i(t)}}{\sum_{i=1}^{N} \frac{\lambda_i}{A_i(t) \sqrt{\beta_i(t)^2 + y^2}}} \exp \left\{ -\frac{1}{2A_i^2(t)} \left[ L_i + \sinh^{-1} \left( \frac{y}{\beta_i(t)} \right) \right]^2 \right\} } \]

The associated option price (mixture of the option prices associated to the basic processes) leads to steep skews:

Figure 5: We set, \( T = 1, N = 2, (A_1(1), A_2(1)) = (0.01, 0.04), (L_1, L_2) = (-0.056, -0.408), \) \( (\lambda_1, \lambda_2) = (0.1, 0.9), \mu = 0 \) and \( S_0 = 0.055. \)