ALTERNATIVE ASSET-PRICE DYNAMICS AND VOLATILITY SMILES

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STYLIZED FACTS MOTIVATING the PAPER

- Traders use the Black-Scholes formula to price plain-vanilla options.
- Options are priced through their *implied volatility*. This is the σ parameter to plug into the Black-Scholes formula to match the corresponding market price:

$$S_0 e^{-qT} \Phi\left(\frac{\ln\frac{S_0}{K} + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - K e^{-rT} \Phi\left(\frac{\ln\frac{S_0}{K} + (r - q - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) = C(K, T)$$

- Implied volatilities vary with strike and maturity. They are skew-shaped (low-strikes vols are higher than high-strikes vols) or smile-shaped (the volatility is minimum around the underlying forward price).

STYLIZED FACTS MOTIVATING the PAPER (cont'd)

Consequences:

- The Black-Scholes model cannot consistently price all options traded in a market (the risk-neutral distribution is not lognormal).
- Need for an alternative asset price model to price exotics or non quoted plain-vanilla options.
- The model should:
  - Feature explicit asset-price dynamics with a known marginal distribution.
  - Imply analytical formulas for European options.
  - Imply a good fitting of market data (reasonable number of parameters).
  - Be stable enough.

MAJOR REFERENCES

We tackle the issue of pricing general volatility structures by assuming a suitable *local volatility* model.

References:

- Brigo, D., Mercurio, F. (2000) A Mixed-up Smile. *Risk*, September, 123-126.
- Brigo, D., Mercurio, F. (2001) Displaced and Mixture Diffusions for Analytically-Tractable Smile Models. In *Mathematical Finance - Bachelier Congress 2000*, Geman, H., Madan, D.B., Pliska, S.R., Vorst, A.C.F., eds. *Springer Finance*, Springer, Berlin.
- Brigo, D., Mercurio, F. (2001) Lognormal-Mixture Dynamics and Calibration to Market Volatility Smiles, International Journal of Theoretical & Applied Finance, forthcoming.
- Brigo, D., Mercurio, F. (2001) *Interest Rate Models: Theory and Practice. Springer Finance*, Springer, Berlin.

The RELATED LITERATURE

FIRST APPROACH: *Alternative Explicit Dynamics*

- It immediately leads to volatility smiles or skews.
- Examples: The general CEV process (Cox (1975) and Cox and Ross (1976)). A general class of processes is due to Carr, Tari and Zariphopoulou (1999).

SECOND APPROACH: *Continuum of Traded Strikes*

- It goes back to Breeden and Litzenberger (1978).
- Examples: Dupire (1994, 1997) and Derman and Kani (1994, 1998).

THIRD APPROACH: *Lattice Approach*

- Based on finding the risk-neutral probabilities in a tree that best fit market prices due to some smoothness criterion.
- Examples: Rubinstein (1994), Jackwerth and Rubinstein (1996) and Britten-Jones and Neuberger (1999).

The RELATED LITERATURE (cont'd)

FOURTH APPROACH: *Incomplete Market*

- Stochastic-volatility models: Hull and White (1987) and Heston (1993) (arbitrary correlation between the asset and its volatility).
- Jump-diffusion models: Merton (1976), Amin (1993) and Prigent, Renault and Scaillet (2000).

FIFTH APPROACH: *Market Model*

- Analogous to the Market Model for interest rates
- Examples: Schonbucher (1998), Ledoit and Santa-Clara (1999) and Brace et al. (2001).

PRICING the SMILE for RISK MANAGEMENT PURPOSES

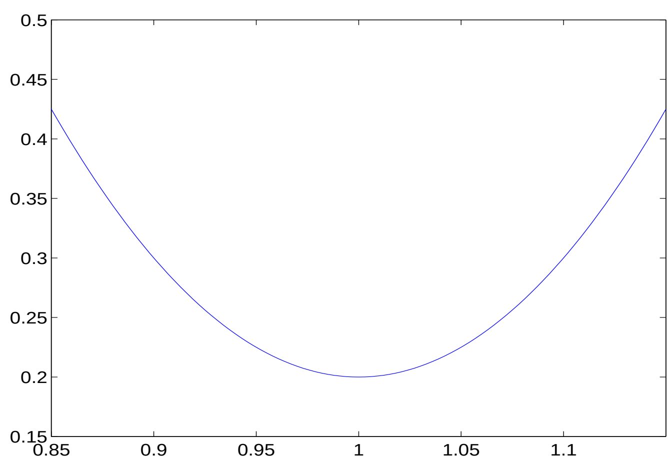
Assume we hold a one-year maturity call with strike 90 and underlying stock price 100. If $r = 0.05$ and $\sigma = 0.3$, the call price is:

$$\text{BS}(S_0 = 100, K = 90, \sigma = 0.3) = 19.697$$

Now assume that the stock price drops to 90. The call price becomes:

$$\text{BS}(S_0 = 90, K = 90, \sigma = 0.3) = 12.808;$$

However, if we take into account the volatility smile:



$$\text{BS}(S_0 = 90, K = 90, \sigma = 0.2) = 9.406;$$

An ANALYTICALLY TRACTABLE CLASS of MODELS

We propose a class of analytically tractable models for an asset-price dynamics that are flexible enough to reproduce a large variety of market volatility structures.

The asset underlies a given option market (needs not be tradable). We can think of an exchange rate, a stock index, and even a forward LIBOR rate.

We assume that:

- The T-forward risk-adjusted measure Q^T exists.
- The dynamics of the asset price S under Q^T is

$$dS_t = \mu S_t dt + \sigma(t, S_t) S_t \, dW_t, \quad S_0 > 0,$$

  where μ is a constant and σ is well behaved.
- The marginal density of S under Q^T is equal to the *weighted average of* the known *densities* of some given diffusion processes.

An ANALYTICALLY TRACTABLE CLASS: the PROBLEM FORMULATION

Let us then consider N diffusion processes with dynamics given by

$$dS_t^i = \mu S_t^i dt + v_i(t, S_t^i)\, dW_t, \quad S_0^i = S_0,$$

where $v_i(t, y)$'s are real functions satisfying regularity conditions to ensure existence and uniqueness of the solution to the SDE.

For each t, we denote by $p_t^i(\cdot)$ the density function of S_t^i ($p_0^i(y)$ is the Dirac-δ function centered in S_0^i).

Problem. Derive the local volatility $\sigma(t, S_t)$ such that the Q^T-density of S_t satisfies

$$p_t(y) := \sum_{i=1}^{N} \lambda_i p_t^i(y),$$

where λ_i's are (strictly) positive constants such that $\sum_{i=1}^{N} \lambda_i = 1$.

An ANALYTICALLY TRACTABLE CLASS of MODELS: the PROBLEM SOLUTION

N.B. $p_t(\cdot)$ is a proper Q^T-density function:

$$\int_0^{+\infty} y p_t(y) dy = \sum_{i=1}^{N} \lambda_i \int_0^{+\infty} y p_t^i(y) dy = \sum_{i=1}^{N} \lambda_i S_0 e^{\mu t} = S_0 e^{\mu t}$$

Solution. Apply the Fokker-Planck equation (FPE)

$$\frac{\partial}{\partial t} p_t(y) = -\frac{\partial}{\partial y} \left(\mu y p_t(y)\right) + \frac{1}{2} \frac{\partial^2}{\partial y^2} \left(\sigma^2(t,y) y^2 p_t(y)\right),$$

to back out σ, given that the FPE holds for each basic process as well.

Applying the definition of $p_t(y)$, the linearity of the derivative operator and the above FPEs, we get:

$$\boxed{dS_t = \mu S_t dt + \sqrt{\frac{\sum_{i=1}^{N} \lambda_i v_i^2(t, S_t) p_t^i(S_t)}{\sum_{i=1}^{N} \lambda_i S_t^2 p_t^i(S_t)}}\, S_t\, dW_t}$$

N.B. This SDE, however, only defines some candidate dynamics leading to the marginal density $p_t(\cdot)$.

An ANALYTICALLY TRACTABLE CLASS of MODELS: OPTION PRICING

Let us give for granted that the previous SDE has a unique strong solution and consider a European option with maturity T, strike K and written on the asset.

The option value at time $t = 0$ is ($\omega = \pm 1$):

$$
\begin{aligned}
\mathcal{O} &= P(0,T)E^T\left\{[\omega(S_T - K)]^+\right\} \\
&= P(0,T)\int_0^{+\infty}[\omega(y-K)]^+\sum_{i=1}^N \lambda_i p_T^i(y)dy \\
&= \sum_{i=1}^N \lambda_i P(0,T)\int_0^{+\infty}[\omega(y-K)]^+ p_T^i(y)dy = \sum_{i=1}^N \lambda_i \mathcal{O}_i
\end{aligned}
$$

Remark [Greeks]. The same convex combination applies also to all option Greeks.

Remark [Why a mixture of densities?] i) if p^i's are analytically tractable, we immediately have closed-form formulas for European options; ii) the number of model parameters is virtually unlimited.

The LOGNORMAL-MIXTURE CASE

We now assume that, for each i,

$$v_i(t, y) = \sigma_i(t) y,$$

where σ_i's are deterministic, continuous and bounded from below by positive constants.

We also assume there exists an $\varepsilon > 0$ such that $\sigma_i(t) = \sigma_0 > 0$, for each t in $[0, \varepsilon]$ and $i = 1, \ldots, N$.

Remark [Why a mixture of lognormals?]

- It is analytically tractable and obviously linked to the Black-Scholes model.
- The log-returns $\ln(S_t/S_0)$, $t > 0$, are more leptokurtic than in the Gaussian case.
- It works well in many practical situations. See Ritchey (1990), Melick and Thomas (1997), Bhupinder (1998), Guo (1998) and Alexander and Narayanan (2001).

The LOGNORMAL-MIXTURE CASE: the ASSET-PRICE DYNAMICS

Proposition. If we set $V_i(t) := \sqrt{\int_0^t \sigma_i^2(u)du}$ and

$$\nu(t,y) = \sqrt{\frac{\sum_{i=1}^N \lambda_i \sigma_i^2(t) \frac{1}{V_i(t)} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{S_0} - \mu t + \frac{1}{2}V_i^2(t)\right]^2\right\}}{\sum_{i=1}^N \lambda_i \frac{1}{V_i(t)} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{S_0} - \mu t + \frac{1}{2}V_i^2(t)\right]^2\right\}}},$$

for $(t,y) > (0,0)$ and $\nu(0,S_0) := \sigma_0$ for, the SDE

$$dS_t = \mu S_t dt + \nu(t,S_t) S_t dW_t,$$

has a unique strong solution whose marginal density is

$$p_t(y) = \sum_{i=1}^N \lambda_i \frac{1}{y V_i(t)\sqrt{2\pi}} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{S_0} - \mu t + \frac{1}{2}V_i^2(t)\right]^2\right\}.$$

N.B. We notice that for $(t,y) > (0,0)$

$$\nu^2(t,y) = \sum_{i=1}^N \Lambda_i(t,y)\, \sigma_i^2(t),$$

where, $\Lambda_i(t,y) \geq 0$ and $\sum_{i=1}^N \Lambda_i(t,y) = 1$. Therefore:

$$0 < \tilde{\sigma} \leq \nu(t,y) \leq \hat{\sigma} < +\infty \quad \text{for each } t, y > 0.$$

The LOGNORMAL-MIXTURE CASE: the ASSET-PRICE DYNAMICS (cont'd)

Proof. The proof is based on the Theorem 12.1 in Section V.12 of Rogers and Williams (1996).

We write $S_t = \exp(Z_t)$, where

$$dZ_t = \left[\mu - \frac{1}{2}\sigma^2\left(t, e^{Z_t}\right)\right] dt + \sigma\left(t, e^{Z_t}\right) dW_t,$$

The coefficients of this SDE are bounded, and hence satisfy the usual linear-growth condition.

Setting $u(t,z) := \sigma(t,e^z)$, we have that $\frac{\partial u^2}{\partial z}(t,z)$ is well defined and continuous for $(t,z) \in (0,M] \times I\!R$, $M > 0$ (continuity of σ_i and V_i), and (u is constant for $t \in [0,\epsilon]$)

$$\lim_{t \to 0} \frac{\partial u^2}{\partial z}(t,z) = 0.$$

The derivative $\frac{\partial u^2}{\partial z}(t,z)$ is thus bounded on each compact set $[0,M] \times [-M,M]$, and so is $\frac{\partial u}{\partial z}(t,z) = \frac{1}{2u(t,z)} \frac{\partial u^2}{\partial z}(t,z)$ since σ is bounded from below. Hence, u is locally Lipschitz.

Theorem 12.1 in Section V.12 of Rogers and Williams (1996)

Suppose that the coefficients σ and b in the SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

are such that for each N there is some K_N such that:

$$|\sigma(s, x) - \sigma(s, y)| \leq K_N|x - y|$$
$$|b(s, x) - b(s, y)| \leq K_N|x - y|$$

whenever $\max(|x|, |y|) \leq N$ and $0 \leq s \leq N$. Suppose also that for each constant $T > 0$, there is some C_T such that, for $0 \leq s \leq T$,

$$|\sigma(s, x)| + |b(s, x)| \leq C_T(1 + |x|)$$

then the above SDE is an exact SDE.

The LOGNORMAL-MIXTURE CASE: OPTION PRICING

Proposition. The time-0 price of a European option with maturity T, strike K and written on the asset is

$$\mathcal{O} = \omega P(0,T) \sum_{i=1}^{N} \lambda_i \left[S_0 e^{\mu T} \Phi\!\left(\omega \frac{\ln \frac{S_0}{K} + \left(\mu + \frac{1}{2}\eta_i^2\right) T}{\eta_i \sqrt{T}} \right) - K \Phi\!\left(\omega \frac{\ln \frac{S_0}{K} + \left(\mu - \frac{1}{2}\eta_i^2\right) T}{\eta_i \sqrt{T}} \right) \right],$$

where $\omega = 1(-1)$ for a call (put), and $\eta_i := \frac{V_i(T)}{\sqrt{T}}$.

For each T, the implied volatility is smile-shaped:

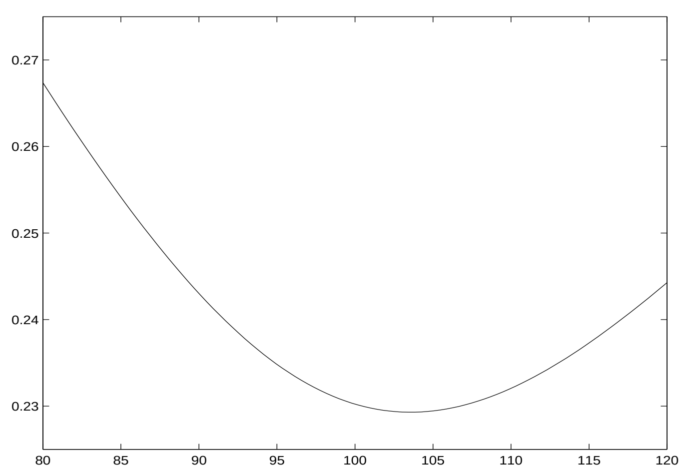


Figure 1: $\mu = .035$, $T = 1$, $(V_1(1), V_2(1), V_3(1)) = (.5, .1, .2)$, $(\lambda_1, \lambda_2, \lambda_3) = (.2, .3, .5)$, $S_0 = 100$.

The LOGNORMAL-MIXTURE CASE: the IMPLIED VOLATILITY

Definition. Defining the moneyness m by

$$m := \ln \frac{S_0}{K} + \mu T,$$

the *Black-Scholes volatility* $\hat{\sigma}(m)$ that, for a given T, is *implied* by the above option price is implicitly given by

$$P(0,T)S_0 e^{\mu T}\left[\Phi\!\left(\frac{m+\frac{1}{2}\hat{\sigma}(m)^2 T}{\hat{\sigma}(m)\sqrt{T}}\right) - e^{-m}\Phi\!\left(\frac{m-\frac{1}{2}\hat{\sigma}(m)^2 T}{\hat{\sigma}(m)\sqrt{T}}\right)\right]$$

$$= P(0,T)S_0 e^{\mu T}\sum_{i=1}^{N}\lambda_i\left[\Phi\!\left(\frac{m+\frac{1}{2}\eta_i^2 T}{\eta_i\sqrt{T}}\right) - e^{-m}\Phi\!\left(\frac{m-\frac{1}{2}\eta_i^2 T}{\eta_i\sqrt{T}}\right)\right]$$

Proposition. The Black-Scholes volatility implied by the above option price is (neglecting $o(m^2)$ terms):

$$\hat{\sigma}(m) = \hat{\sigma}(0) + \frac{1}{2\hat{\sigma}(0)T}\sum_{i=1}^{N}\lambda_i\left[\frac{\hat{\sigma}(0)}{\eta_i}e^{\frac{1}{8}\left(\hat{\sigma}(0)^2-\eta_i^2\right)T} - 1\right]m^2$$

where the ATM-forward implied volatility $\hat{\sigma}(0)$ is

$$\hat{\sigma}(0) = \frac{2}{\sqrt{T}}\Phi^{-1}\left(\sum_{i=1}^{N}\lambda_i\Phi\left(\frac{1}{2}\eta_i\sqrt{T}\right)\right).$$

SHIFTING the DISTRIBUTION

Let us define a new asset-price process A by:

$$A_t = A_0 \alpha e^{\mu t} + S_t,$$

where α is a real constant.

By Ito's formula, we immediately obtain:

$$dA_t = \mu A_t dt + \nu\big(t, A_t - A_0 \alpha e^{\mu t}\big)\,(A_t - A_0 \alpha e^{\mu t}) dW_t.$$

Some possible densities

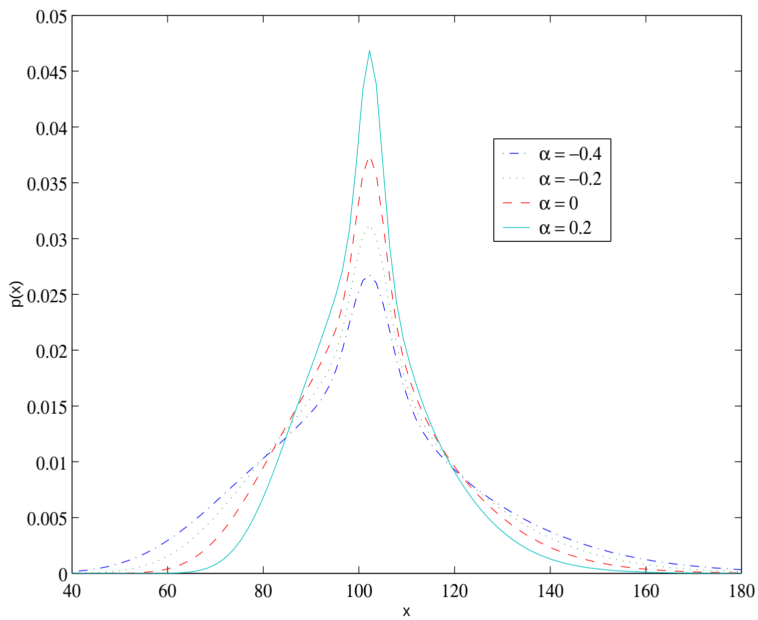


Figure 2: The density function $p(x)$ of A_T for the different values of $\alpha \in \{-0.4, -0.2, 0, 0.2\}$, where we set $A_0 = 100$, $\mu = 0.05$, $T = 0.5$, $N = 3$, $(\eta_1(T), \eta_2(T), \eta_3(T)) = (0.25{,}0.09{,}0.04)$ and $(\lambda_1, \lambda_2, \lambda_3) = (0.8{,}0.1{,}0.1)$.

SHIFTING the DISTRIBUTION: OPTION PRICING

Proposition. The time-0 price of a European option with strike K, maturity T and written on the asset is

$$\mathcal{O} = \omega P(0,T) \sum_{i=1}^{N} \lambda_i \left[\mathcal{A}_0 e^{\mu T} \Phi\!\left(\omega \frac{\ln \frac{\mathcal{A}_0}{\mathcal{K}} + \left(\mu + \frac{1}{2}\eta_i^2\right) T}{\eta_i \sqrt{T}} \right) - \mathcal{K} \Phi\!\left(\omega \frac{\ln \frac{\mathcal{A}_0}{\mathcal{K}} + \left(\mu - \frac{1}{2}\eta_i^2\right) T}{\eta_i \sqrt{T}} \right) \right],$$

where $\mathcal{K} = K - A_0 \alpha e^{\mu T}$, $\mathcal{A}_0 = A_0(1-\alpha)$. Moreover:

$$\hat{\sigma}(m) = \hat{\sigma}(0) + \alpha \frac{\sum_{i=1}^{N} \lambda_i \Phi\left(-\frac{1}{2}\eta_i \sqrt{T}\right) - \frac{1}{2}}{\frac{\sqrt{T}}{\sqrt{2\pi}} e^{-\frac{1}{8}\hat{\sigma}(0)^2 T}} m$$

$$+ \frac{1}{2}\left[\frac{1}{T(1-\alpha)} \sum_{i=1}^{N} \frac{\lambda_i}{\eta_i} e^{\frac{1}{8}\left(\hat{\sigma}(0)^2 - \eta_i^2\right)T} - \frac{1}{\hat{\sigma}(0)T} + \frac{\alpha^2}{4}\hat{\sigma}(0) T \left(\frac{\sum_{i=1}^{N} \lambda_i \Phi\left(-\frac{1}{2}\eta_i \sqrt{T}\right) - \frac{1}{2}}{\frac{\sqrt{T}}{\sqrt{2\pi}} e^{-\frac{1}{8}\hat{\sigma}(0)^2 T}} \right)^2 \right] m^2 + o(m^2)$$

where the ATM-forward implied volatility $\hat{\sigma}(0)$ is now

$$\hat{\sigma}(0) = \frac{2}{\sqrt{T}} \Phi^{-1}\left((1-\alpha) \sum_{i=1}^{N} \lambda_i \Phi\left(\frac{1}{2}\eta_i \sqrt{T} \right) + \frac{\alpha}{2} \right).$$

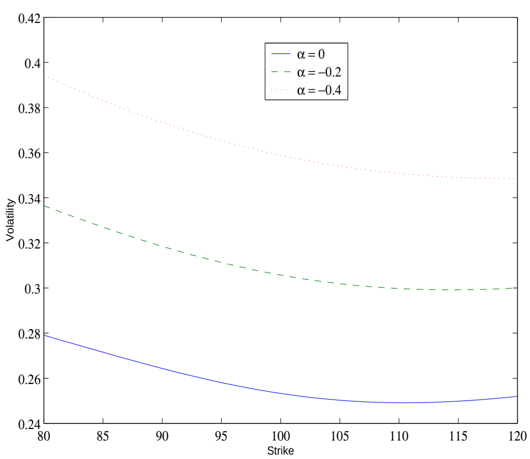
SHIFTING the DISTRIBUTION: the IMPACT of α

Decreasing α, the variance of the asset-price at each time increases while maintaining the correct expectation:

$$E(A_t) = A_0 e^{\mu t}$$

$$\text{Var}(A_t) = A_0^2 (1-\alpha)^2 e^{2\mu t} \left(\sum_{i=1}^{N} \lambda_i e^{V_i^2(t)} - 1 \right).$$

- α concurs to determine the implied-volatility level.
- α moves the strike where the volatility is minimum.



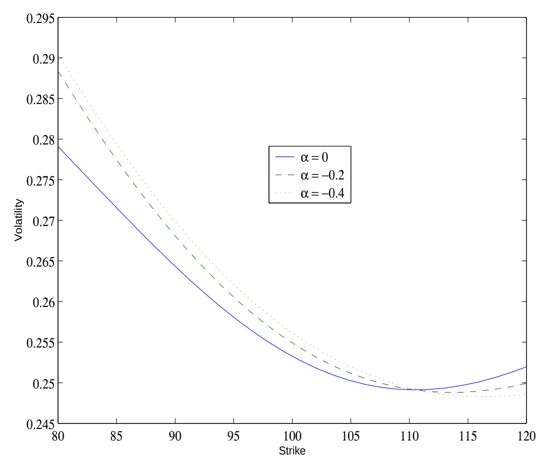


Figure 3: $A_0 = 100$, $\mu = 0.05$, $N = 2$, $T = 2$, $K \in [80, 120]$. Left: implied volatility curve for $(\eta_1(T), \eta_2(T)) = (0.35, 0.1)$, $(\lambda_1, \lambda_2) = (0.6, 0.4)$, $\alpha \in \{0, -0.2, -0.4\}$; Right: implied volatility curve for $(\lambda_1, \lambda_2) = (0.6, 0.4)$ in the three cases: a) $(\eta_1(T), \eta_2(T)) = (0.35, 0.1)$, $\alpha = 0$; b)$(\eta_1(T), \eta_2(T)) = (0.110, 0.355)$, $\alpha = -0.2$; c)$(\eta_1(T), \eta_2(T)) = (0.098, 0.298)$, $\alpha = -0.4$;

APPLYING the MODEL in PRACTICE

The asset is a stock/index.

- Assume constant interest rates (all equal to $r > 0$), and set $\mu = r - q$ (q is the dividend yield).
- Pronounced skews can be produced (but not highly steep curves for very short maturities).

The asset is an exchange rate.

- Assume constant interest rates, and set $\mu = r - r_f$ (r_f is the foreign risk-free rate).
- Exchange rate volatilities are typically smile-shaped.

The asset is a forward LIBOR rate.

- The forward LIBOR rate at time t for $[S, T]$ is
$$F(t, S, T) = \frac{1}{\tau(S, T)} \left[\frac{P(t, S)}{P(t, T)} - 1 \right],$$
where $\tau(S, T)$ is the year fraction from S to T.
- Since $F(\cdot, S, T)$ is a martingale under Q^T, set $\mu = 0$.

The CALIBRATION to MARKET DATA

General comments

- The (virtually unlimited) number of parameters can render the calibration to market data very accurate.
- When minimizing the "distance" between model and market prices, the search for a global minimum can be cumbersome and inevitably slow.
- The use of a local-search algorithm speeds up the calibration process (in practice, it is advisable to combine the two searches).

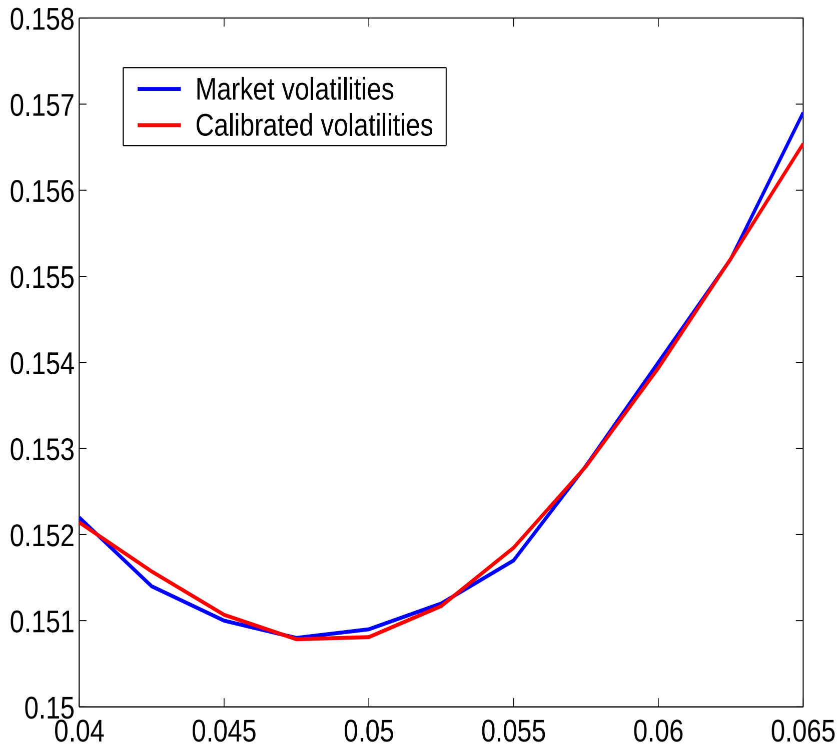
Getting ready for a calibration

- Assume we have M option maturities $T_1 < \ldots < T_M$.
- Set $v_{i,j} := \eta_i(T_j)$ for $i = 1, \ldots, N$ and $j = 1, \ldots, M$.
- Impose the constraints $v_{i,j+1} > v_{i,j}\sqrt{T_j/T_{j+1}}$, $\forall i, j$.
- More constraints: $K > A_0 \alpha e^{\mu T}$, for each "traded" K.
- We may assume that $v_{i,j} = \bar{v}_i > 0$.
- It is enough to set $N = 2, 3$ in most applications.

FIRST EXAMPLE of CALIBRATION to REAL MARKET DATA

Data: two-year Euro caplet volatilities as of November 14th, 2000 (LIBOR resetting at 1.5 years).

We set: $N = 2$, $v_i := \eta_i(1.5)$, $i = 1, 2$, $\lambda_2 = 1 - \lambda_1$. We minimize the squared percentage difference between model and market (mid) prices. We get: $\lambda_1 = 0.241$, $\lambda_2 = 0.759$, $v_1 = 0.125$, $v_2 = 0.194$, $\alpha = 0.147$.



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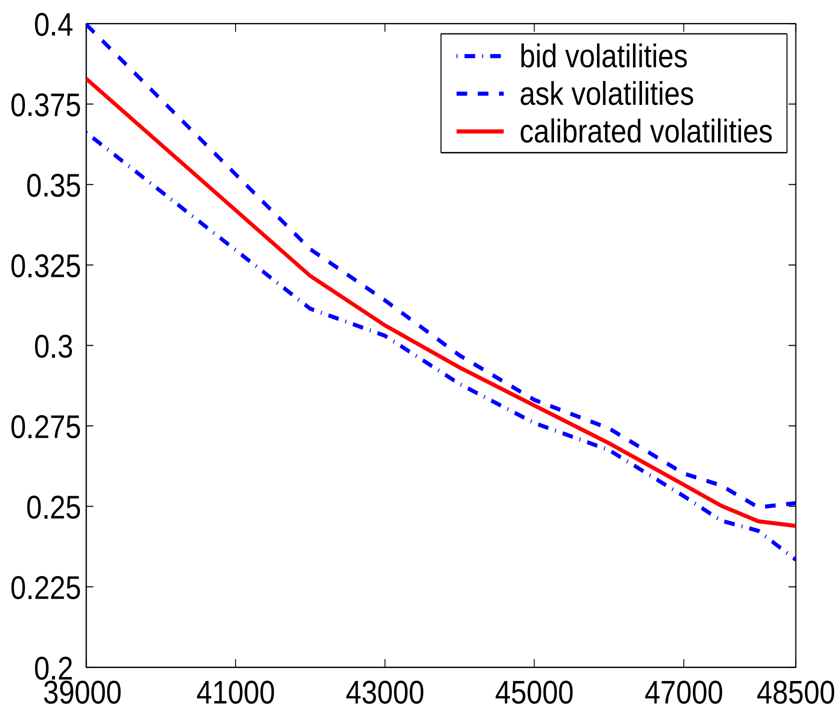
SECOND EXAMPLE of CALIBRATION to REAL MARKET DATA

Data: Italian MIB30 equity index on March 29, 2000, at 3,21pm (most liquid puts with the shortest maturity).

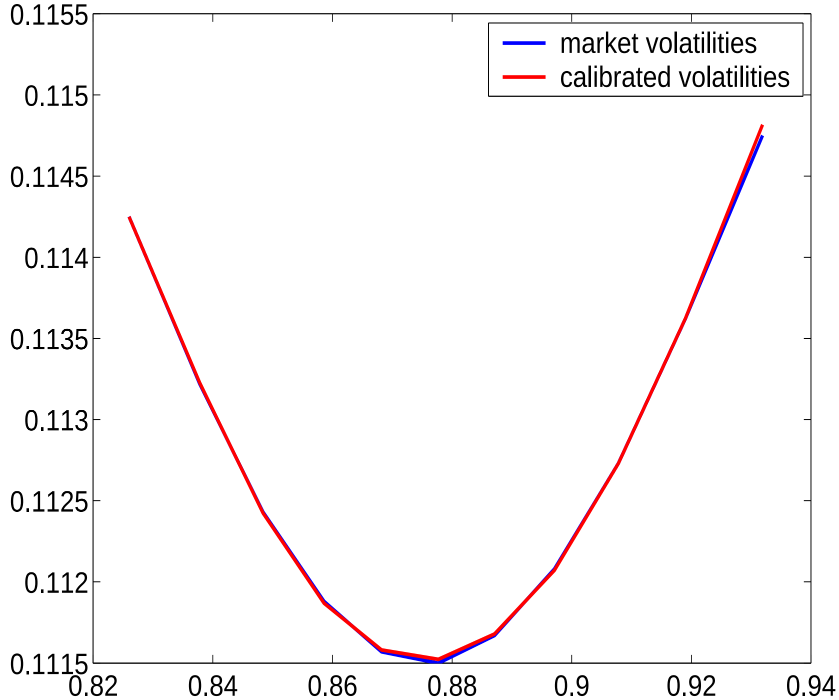
We set $N = 3$, $v_i := \eta_i(T)$ $(i = 1, 2, 3)$, $\lambda_3 = 1 - \lambda_1 - \lambda_2$. We minimize the squared percentage difference between model and market mid prices. We get: $\lambda_1 = 0.201$, $\lambda_2 = 0.757$, $v_1 = 0.019$, $v_2 = 0.095$, $v_3 = 0.229$, $\alpha = -1.852$.



THIRD EXAMPLE of CALIBRATION to REAL MARKET DATA

Data: USD/Euro two-month implied volatilities as of May 21, 2001.

We set $N = 2$, $v_i := \eta_i(0.167)$ $(i = 1, 2)$, $\lambda_2 = 1 - \lambda_1$. We minimize the squared percentage difference between model and market mid prices. We get: $\lambda_1 = 0.451$, $v_1 = 0.129$, $v_2 = 0.114$, $\alpha = 0.076$.



APPLICATION to the FORWARD-LIBOR MARKET MODEL

Proposition. The dynamics of $F_k := F(\cdot; T_{k-1}, T_k)$ under the forward measure Q^i in the three cases $i < k$, $i = k$ and $i > k$ are, respectively,

$i < k, \; t \leq T_i :$

$$dF_k(t) = \nu_k(t, F_k(t)) F_k(t) \sum_{j=i+1}^{k} \frac{\rho_{k,j} \tau_j \nu_j(t, F_j(t)) F_j(t)}{1 + \tau_j F_j(t)} \, dt + \nu_k(t, F_k(t)) F_k(t) \, dZ_k(t),$$

$i = k, \; t \leq T_{k-1} :$

$$dF_k(t) = \nu_k(t, F_k(t)) F_k(t) \, dZ_k(t),$$

$i > k, \; t \leq T_{k-1} :$

$$dF_k(t) = -\nu_k(t, F_k(t)) F_k(t) \sum_{j=k+1}^{i} \frac{\rho_{k,j} \tau_j \nu_j(t, F_j(t)) F_j(t)}{1 + \tau_j F_j(t)} \, dt + \nu_k(t, F_k(t)) F_k(t) \, dZ_k(t),$$

where $Z = Z^i$ is a Brownian motion under Q^i. All the above equations admit a unique strong solution.

FURTHER EXTENSIONS: a LOGNORMAL MIXTURE with DIFFERENT MEANS

Let us consider the instrumental processes

$$dS_t^i = \mu_i(t) S_t^i dt + \sigma_i(t) S_t^i dW_t, \quad S_0^i = S_0,$$

and look for a diffusion coefficient $\psi(\cdot,\cdot)$ such that

$$dS_t = \mu S_t dt + \psi(t, S_t) S_t dW_t$$

has a solution with marginal density $p_t(y) = \sum_{i=1}^m \lambda_i p_t^i(y)$.

Denote by $\nu(t, S_t)$ the solution of the analogous problem when the basic processes have the same drift μ:

$$\nu(t,y)^2 = \frac{\sum_{i=1}^m \lambda_i \sigma_i(t)^2 p_t^i(y)}{\sum_{i=1}^m \lambda_i p_t^i(y)}$$

It is then possible to show that

$$\psi(t,y)^2 := \nu(t,y)^2 + \frac{2\sum_{i=1}^m \lambda_i(\mu_i(t) - \mu)\int_y^{+\infty} x p_t^i(x) dx}{y^2 \sum_{i=1}^m \lambda_i p_t^i(y)}$$

It is also possible to prove that $\psi(\cdot,\cdot)$ has linear growth and does not explode in finite time.

FURTHER EXTENSIONS: the CASE of HYPERBOLIC-SINE BASIC PROCESSES

Consider now the instrumental processes, $i = 1, \dots, N$,

$$S_i(t) = \beta_i(t) \sinh\left[\int_0^t \alpha_i(u) dW_u - L_i\right], \quad S_i(0) = S_0$$

where α_i's are positive functions, L_i's are negative constants and

$$\beta_i(t) = \frac{S_0\, e^{\mu t - \frac{1}{2}\int_0^t \alpha_i^2(u)du}}{\sinh(-L_i)}\,.$$

N.B. S_i is an increasing function of a time-changed Brownian motion.

The SDE followed by each S_i is given by

$$dS_i(t) = \mu S_i(t)dt + \alpha_i(t)\sqrt{\beta_i^2(t) + S_i^2(t)}\, dW_t.$$

Setting $A_i(t) := \sqrt{\int_0^t \alpha_i^2(u)du}$, the time-$t$ marginal density of S_i is

$$p_t^i(y) = \frac{\exp\left\{-\frac{1}{2A_i^2(t)}\left[L_i + \sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right]^2\right\}}{A_i(t)\sqrt{2\pi}\sqrt{\beta_i^2(t) + y^2}}.$$

The CASE of HYPERBOLIC-SINE BASIC PROCESSES (cont'd)

A straightforward integration leads to the call price:

$$\mathrm{C}(T,K)=P(0,T)\left[\frac{S_0\,e^{\mu T}}{2\sinh(-L_i)}\Big(e^{-L_i}\Phi[\bar{y}_i(T)+A_i(T)] - e^{L_i}\Phi[\bar{y}_i(T)-A_i(T)]\Big) - K\Phi\big(\bar{y}_i(T)\big)\right],$$

where we set

$$\bar{y}_i(T) := -\frac{L_i}{A_i(T)} - \frac{1}{A_i(T)}\sinh^{-1}\left(\frac{K}{\beta_i(T)}\right).$$

This price leads to steeply decreasing implied volatilities:

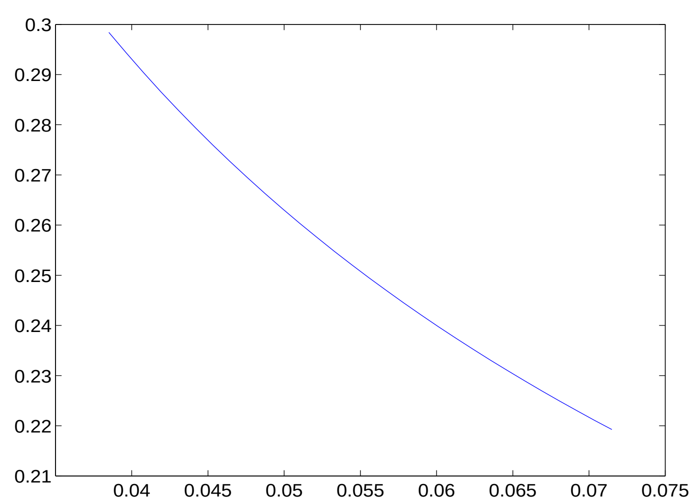


Figure 4: We set, $T=1$, $A_1(1)=0.01$, $L_1=-0.05$, $\mu=0$ and $S_0=0.055$.

The CASE of HYPERBOLIC-SINE BASIC PROCESSES: OPTION PRICING

The previous general results on densities-mixture dynamics immediately yield the following SDE:

$$dS(t) = \mu S(t)dt + \chi(t, S(t))\, dW_t$$

$$\chi(t,y) = \sqrt{\frac{\sum_{i=1}^{N} \lambda_i \frac{\alpha_i^2(t)\sqrt{\beta_i(t)^2+y^2}}{A_i(t)} \exp\left\{-\frac{1}{2A_i^2(t)}\left[L_i + \sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right]^2\right\}}{\sum_{i=1}^{N} \frac{\lambda_i}{A_i(t)\sqrt{\beta_i(t)^2+y^2}} \exp\left\{-\frac{1}{2A_i^2(t)}\left[L_i + \sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right]^2\right\}}}$$

The associated option price (mixture of the option prices associated to the basic processes) leads to steep skews:

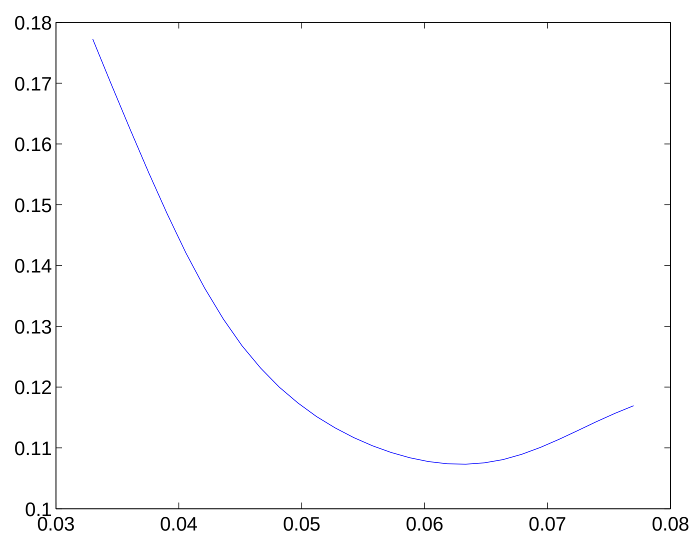


Figure 5: We set, $T = 1$, $N = 2$, $(A_1(1), A_2(1)) = (0.01, 0.04)$, $(L_1, L_2) = (-0.056, -0.408)$, $(\lambda_1, \lambda_2) = (0.1, 0.9)$, $\mu = 0$ and $S_0 = 0.055$.