Alternative Asset-Price Dynamics and Volatility Smile [†]

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Abstract

We propose a general class of analytically tractable models for the dynamics of an asset price leading to smiles or skews in the implied volatility structure. The considered asset can be an exchange rate, a stock index, or even a forward Libor rate. The class is based on an explicit SDE under a given forward measure. The models we propose feature i) explicit asset-price dynamics, ii) virtually unlimited number of parameters, iii) analytical formulas for European options.

We then consider the fundamental case where the asset price density is given, at every time, by a mixture of lognormal densities. We also derive an explicit approximation of the implied volatility function in terms of the option moneyness. We finally introduce two other examples: the first is still based on lognormal densities, but it allows for different means in the distributions; the second is instead based on processes of hyperbolic-sine type.

Keywords

Stock-Price Dynamics, Analytical Tractability, Explicit Option Pricing, Fokker-Planck Equation, Mixture of Densities, Lognormal Densities, Hyperbolic-Sine Processes, Volatility Smile.

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1 Introduction

It is widely known that the Black and Scholes (1973) assumption of a constant volatility for pricing derivative securities with the same underlying asset fails to hold true in practice. In fact, one commonly observes that the term structure of implied volatilities features some particular shapes that are termed "skews" and "smiles". The term skew is used to indicate those structures where, for a fixed maturity, low-strikes implied-volatilities are higher than high-strikes implied-volatilities. The term smile is used instead to denote those structures where, again for a fixed maturity, the volatility has a minimum value around the underlying forward price.

If the implied volatilities for different strikes were equal for each fixed maturity (but different for different maturities), a simple extension of the Black-Scholes model would exactly reproduce the market option prices: one has just to introduce a time-dependent (deterministic) volatility σ_t in the Black-Scholes dynamics for the asset price

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t,$$

and, given the N increasing maturities $T_1, \ldots, T_N$, to recursively solve

$$\int_0^{T_i} \sigma_t^2 dt = v_i^2 T_i,$$

where v_i is the implied volatility for the maturity T_i.[1] Unfortunately, this extended Black-Scholes model can not describe options data in a satisfactory way, because more complex volatility structures are present in real financial markets. This issue can then be tackled by introducing a more articulated form of the volatility coefficient in the asset-price dynamics. This is the approach we follow in this paper. We in fact propose several asset-price models by specifying the asset price dynamics under a specific forward measure. The volatility σ_t we introduce is a function of time t and of the asset price S_t at the same time.

Several works have tried to address the problem of a good, possibly exact, fitting of market option data. We now briefly review the major approaches that have been proposed.

A first approach is based on assuming an *alternative explicit dynamics* for the asset-price process that immediately leads to volatility smiles or skews. In general this approach does not provide sufficient flexibility to properly calibrate the whole volatility surface. Examples are the general CEV process of Cox (1975) and Cox and Ross (1976) and the hyperbolic diffusion model of Bibby and Sørensen (1997). A general class of processes is due to Carr et al. (1999).

A second approach is based on the assumption of a *continuum of traded strikes* and goes back to Breeden and Litzenberger (1978). Successive developments are due to Dupire (1994, 1997), Derman and Kani (1994, 1998) and Dempster and Richard (1999) who derive an explicit expression for the Black-Scholes volatility as a function of strike and maturity.

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[1] In principle, we may get imaginary values for σ_t. However, if the term structure of implied volatilities is sufficiently smooth, this problem is not encountered.

This approach has the major drawback that one needs to smoothly interpolate option prices between consecutive strikes in order to be able to differentiate them twice with respect to the strike. Explicit expressions for the risk-neutral asset price dynamics are also derived by Avellaneda et al. (1997) by minimizing the relative entropy to a prior distribution, and by Brown and Randall (1999) by assuming a quite flexible analytical function describing the volatility surface.

Another approach, pioneered by Rubinstein (1994), consists in finding the risk-neutral probabilities in a binomial/trinomial model for the asset price that lead to a best fitting of market option prices due to some smoothness criterion. We refer to this approach as to the *lattice* approach. Further examples are in Jackwerth and Rubinstein (1996) and Andersen and Brotherton-Ratcliffe (1997) who use instead finite-difference grids. A different lattice approach is due to Britten-Jones and Neuberger (1999).

A further approach is given by what we may refer to as *incomplete-market* approach. It includes stochastic-volatility models, such as those of Hull and White (1987), Heston (1993) and Tompkins (2000a, 2000b),[2] and jump-diffusion models, such as those of Merton (1976) and Prigent, Renault and Scaillet (2001).

A last approach is based on the so called *market model for implied volatility*. The first examples are in Schönbucher(1999) and Ledoit and Santa Clara (1998). A recent application in case of the forward Libor market model is due to Brace et al. (2001).

In general the problem of finding a risk-neutral distribution that consistently prices all quoted options is largely undetermined. A possible solution is given by assuming a particular *parametric risk-neutral distribution* depending on several, possibly time-dependent, parameters and then use such parameters for the volatility calibration.[3] By applying an approach similar to that of Dupire (1994, 1997), we address this question and find dynamics leading to parametric risk-neutral distributions that are flexible enough for practical purposes. The resulting processes combine therefore the *parametric risk-neutral distribution* approach with the *alternative dynamics approach*, providing explicit dynamics that lead to flexible parametric risk-neutral densities.

The major challenge our class of models is fit to face is the introduction of a forward-measure distribution that features i) *analytical formulas* for European options, so that the calibration to market data and the computation of Greeks can be extremely rapid, ii) *high number of model parameters*, so as to imply a satisfactory fitting of market data, iii) *explicit asset-price dynamics*, so that exotic claims can be priced through a Monte Carlo simulation.[4]

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[2] In the context of the forward Libor market model, we must also mention the recent work of Rebonato (2001).

[3] An example of this approach is due to Shimko (1993).

[4] Alternative methods of extracting a risk-neutral distribution from option prices are in Malz (1997) and Pirkner et al. (1999). Alternative models with explicit formulas for European options have been proposed by Li (1998) and Bouchouev (2000).

The paper is structured as follows. Section 2 proposes a general class of analytical asset-price models whose associated density is the mixture of some given densities. The example of a mixture of lognormal distributions is then considered in Section 3. Section 4 extends the previous result to case of lognormal densities with different means. Section 5 considers a further example based on basic processes of hyperbolic-sine type. Section 6 concludes the paper.

2 A class of analytically tractable models allowing for volatility smiles

Brigo and Mercurio (2001a) proposed a class of analytically tractable models for an asset-price dynamics that are flexible enough to recover a large variety of market volatility structures. The considered asset underlies a given option market and, as such, needs not be tradable itself. Indeed, we can think of an exchange rate, a stock index, or even a forward Libor rate, since caps and floors are nothing but options on Libor rates.

The diffusion processes they obtained follow from assuming a particular distribution for the asset price S under a specific measure. In order to properly introduce the fundamental examples developed in the next sections, we now quickly review their main results.

We fix a time T and denote by $P(0,T)$ the price at time 0 of a zero-coupon bond with maturity T. We then assume that the T-forward risk-adjusted measure Q^T exists and that the marginal density of S under Q^T is equal to the weighted average of the known densities of some given diffusion processes. This is equivalent to view S as a process whose density at time t coincides with a basic density with probability given by the corresponding weight. The reason for imposing such a marginal density is to achieve analytical tractability with no computational efforts, as formula (12) below will easily show.

Let the dynamics of the asset price S under the forward measure Q^T be expressed by

$$dS_t = \mu S_t dt + \sigma(t, S_t) S_t dW_t, \tag{1}$$

where μ is a constant, W is a Q^T-standard Brownian motion and σ is a well-behaved deterministic function.

The μ parameter is completely specified by the definition of Q^T. In fact, if the asset is a stock paying a continuous dividend yield q and rates are deterministic, then $\mu = r - q$, where r is the time T (continuously compounded) risk-free rate. If the asset is an exchange rate and rates are deterministic, then $\mu = r - r_f$, where r_f is the foreign risk-free rate for the maturity T. If the asset is a forward Libor rate spanning the interval $[T_0, T]$, $T_0 < T$, then $\mu = 0$ due to the martingale property of forward rates under their corresponding measure. Notice, moreover, that under deterministic interest rates, the (assumed unique) risk-neutral measure coincides with each of the possible forward measures.

The function σ, which is usually termed *local volatility* in the financial literature, must be chosen so as to grant a unique strong solution to the SDE (1). In particular, we assume that $\sigma(\cdot,\cdot)$ satisfies, for a suitable positive constant L, the linear-growth condition

$$\sigma^2(t,y)y^2 \le L(1+y^2) \ \text{ uniformly in } t, \tag{2}$$

which basically prevents from explosion in finite time.

Let us then consider N diffusion processes with dynamics given by

$$dS_t^i = \mu S_t^i dt + v_i(t,S_t^i)dW_t, \quad i=1,\dots,N, \quad S_0^i = S_0 \ , \tag{3}$$

with common initial value S_0, and where $v_i(t,y)$'s are real functions satisfying regularity conditions to ensure existence and uniqueness of the solution to the SDE (3). In particular we assume that, for suitable positive constants L_i's, the following linear-growth conditions hold:

$$v_i^2(t,y) \le L_i(1+y^2) \ \text{ uniformly in } t, \quad i=1,\dots,N. \tag{4}$$

For each t, we denote by $p_t^i(\cdot)$ the density function of S_t^i, i.e., $p_t^i(y) = d(Q^T\{S_t^i \le y\})/dy$, where, in particular, p_0^i is the δ-Dirac function centered in S_0^i.

The problem addressed by Brigo and Mercurio (2001a) is the derivation of the local volatility $\sigma(t,S_t)$ such that the Q^T-density of S satisfies, for each time t,

$$p_t(y) := \frac{d}{dy}\ Q^T\{S_t \le y\} = \sum_{i=1}^N \lambda_i \frac{d}{dy}\ Q^T\{S_t^i \le y\} = \sum_{i=1}^N \lambda_i p_t^i(y), \tag{5}$$

where the λ_i's are strictly positive constants such that $\sum_{i=1}^N \lambda_i = 1$. Notice that $p_t(\cdot)$ is a proper Q^T-density function since, by definition,

$$\int_0^{+\infty} y p_t(y)dy = \sum_{i=1}^N \lambda_i \int_0^{+\infty} y p_t^i(y)dy = \sum_{i=1}^N \lambda_i S_0 e^{\mu t} = S_0 e^{\mu t}.$$

As in Dupire (1997), the volatility coefficient σ is found by solving the Fokker-Planck equation

$$\frac{\partial}{\partial t}p_t(y) = -\frac{\partial}{\partial y}\left(\mu y p_t(y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(t,y)y^2 p_t(y)\right), \tag{6}$$

given that each density $p_t^i(y)$ satisfies the Fokker-Planck equation

$$\frac{\partial}{\partial t}p_t^i(y) = -\frac{\partial}{\partial y}\left(\mu y p_t^i(y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(v_i^2(t,y) p_t^i(y)\right). \tag{7}$$

After straightforward calculations, Brigo and Mercurio (2001a) obtained that the expression for $\sigma(t,y)$ that is consistent with the marginal density (5) and with the regularity constraint (2) is, for $(t,y) > (0,0)$,

$$\sigma(t,y) = \sqrt{\frac{\sum_{i=1}^N \lambda_i v_i^2(t,y)p_t^i(y)}{\sum_{i=1}^N \lambda_i y^2 p_t^i(y)}}\ , \tag{8}$$

whose square can be characterized as

$$\sigma^2(t,y) = \sum_{i=1}^{N} \Lambda_i(t,y) \frac{v_i^2(t,y)}{y^2}, \tag{9}$$

where we set, for each $i = 1, \ldots, N$ and $(t,y) > (0,0)$,

$$\Lambda_i(t,y) := \frac{\lambda_i p_t^i(y)}{\sum_{i=1}^{N} \lambda_i p_t^i(y)}. \tag{10}$$

The square of the local volatility σ can thus be written as a (stochastic) convex combination of the squared volatilities of the basic processes (3), since $\Lambda_i(t,y) \geq 0$ for each i and (t,y), and $\sum_{i=1}^{N} \Lambda_i(t,y) = 1$.

The function σ may be extended to the semi-axes $\{(t,0) : t > 0\}$ and $\{(0,y) : y > 0\}$ according to the specific choice of the basic densities $p_t^i(\cdot)$.

Formula (8) leads to the following SDE for the asset price under the forward measure Q^T:

$$dS_t = \mu S_t dt + \sqrt{\frac{\sum_{i=1}^{N} \lambda_i v_i^2(t,S_t) p_t^i(S_t)}{\sum_{i=1}^{N} \lambda_i S_t^2 p_t^i(S_t)}} S_t dW_t. \tag{11}$$

Similarly to what happens in the general Dupire's (1997) approach, this SDE, however, must be regarded as defining some candidate dynamics that leads to the marginal density (5). In fact, the conditions we have imposed so far are not sufficient to grant existence and uniqueness of a strong solution, so that a verification must be done on a case-by-case basis.

We are now in a position to fully understand the assumption that the asset marginal density is given by the mixture of known basic densities. When proposing alternative asset-price dynamics, the derivation of closed-form formulas for European options is usually quite problematic. Here, instead, such a problem can be easily avoided by starting from analytically-tractable densities p^i. In fact, assuming that the SDE (11) has a unique strong solution,[5] the time-0 price of a European option with maturity T, strike K and written on the asset is immediately given by

$$\begin{aligned} \mathcal{O}(K,T,\omega) &= P(0,T) E^T \left\{ [\omega(S_T - K)]^+ \right\} \\ &= P(0,T) \int_0^{+\infty} [\omega(y-K)]^+ \sum_{i=1}^{N} \lambda_i p_T^i(y) dy \\ &= \sum_{i=1}^{N} \lambda_i \mathcal{O}_i(K,T,\omega), \end{aligned} \tag{12}$$

where $\omega = 1$ for a call and $\omega = -1$ for a put, E^T denotes expectation under Q^T and $\mathcal{O}_i$ denotes the option price associated with (3).

[5]Two fundamental cases where this assumption holds will be considered in the next sections.

Brigo and Mercurio (2001a) also noticed that, due to the linearity of the derivative operator, the same convex combination applies to all option Greeks. Moreover, since N is arbitrary, the number of parameters that can be introduced in the dynamics and used for a better calibration to market data is virtually unlimited.

3 The mixture-of-lognormals case

Let us now consider the particular case where the densities p_t^i's are all lognormal. Precisely, we assume that, for each i,

$$v_i(t,y) = \sigma_i(t)y, \tag{13}$$

where all σ_i's are deterministic functions of time defined on the interval $[0, T^*]$, $T^* > 0$ a given time horizon. Then, the marginal density of S_t^i conditional on S_0 is given by

$$\begin{aligned} p_t^i(y) &= \frac{1}{yV_i(t)\sqrt{2\pi}} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln \frac{y}{S_0} - \mu t + \tfrac{1}{2}V_i^2(t) \right]^2 \right\}, \\ V_i(t) &:= \sqrt{\int_0^t \sigma_i^2(u)du}. \end{aligned} \tag{14}$$

The reason for considering such basic densities is due to their analytical tractability and obvious connection with the Black and Scholes (1973) model. Moreover, mixtures of lognormal densities turn out to work well in practice when used to reproduce market volatility structures. We mention for instance the works of Ritchey (1990)[6], Melick and Thomas (1997), Bhupinder (1998) and Guo (1998). However, these are mainly empirical works where the assumption of a lognormal-mixture risk-neutral density is introduced for a satisfactory (static) calibration to options data.

Brigo and Mercurio (2001a) developed the asset-price model based on a lognormal-mixture risk-neutral density and derived the asset-price (diffusion) dynamics that implies the given distribution. In this section, we review their major result.

Proposition 3.1 (Brigo and Mercurio (2001a)). *Let us assume that each σ_i is continuous and bounded from below by a positive constant, and that there exists an $\varepsilon > 0$ such that $\sigma_i(t) = \sigma_0 > 0$, for each t in $[0, \varepsilon]$ and $i = 1, \ldots, N$. Then, if we set*

$$\nu(t,y) = \sqrt{\frac{\sum_{i=1}^N \lambda_i \sigma_i^2(t) \frac{1}{V_i(t)} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln \frac{y}{S_0} - \mu t + \frac{1}{2}V_i^2(t) \right]^2 \right\}}{\sum_{i=1}^N \lambda_i \frac{1}{V_i(t)} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln \frac{y}{S_0} - \mu t + \frac{1}{2}V_i^2(t) \right]^2 \right\}}}, \tag{15}$$

[6]Indeed, Ritchey (1990) assumed a mixture of normal densities for the density of the asset log-returns. However, it can be easily shown that this is equivalent to assuming a mixture of lognormal densities for the density of the asset price.

for $(t, y) > (0, 0)$ *and* $\nu(t, y) = \sigma_0$ *for* $(t, y) = (0, S_0)$, *the SDE*

$$dS_t = \mu S_t dt + \nu(t, S_t) S_t dW_t \tag{16}$$

has a unique strong solution whose marginal density is given by the mixture of lognormals

$$p_t(y) = \sum_{i=1}^{N} \lambda_i \frac{1}{y V_i(t) \sqrt{2\pi}} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln \frac{y}{S_0} - \mu t + \tfrac{1}{2} V_i^2(t) \right]^2 \right\}. \tag{17}$$

Moreover, for $(t, y) > (0, 0)$, *we can write*

$$\nu^2(t, y) = \sum_{i=1}^{N} \Lambda_i(t, y) \sigma_i^2(t), \tag{18}$$

where, for each (t, y) *and* i, $\Lambda_i(t, y)$ *is defined in (10) through (14),* $\Lambda_i(t, y) \geq 0$ *and* $\sum_{i=1}^{N} \Lambda_i(t, y) = 1$. *As a consequence*

$$0 < \tilde{\sigma} \leq \nu(t, y) \leq \hat{\sigma} < +\infty \quad \textit{for each } t, y > 0, \tag{19}$$

where

$$\tilde{\sigma} := \inf_{t \geq 0} \left\{ \min_{i=1,\ldots,N} \sigma_i(t) \right\},$$
$$\hat{\sigma} := \sup_{t \geq 0} \left\{ \max_{i=1,\ldots,N} \sigma_i(t) \right\}.$$

The function $\nu(t, y)$ can be extended by continuity to the semi-axes $\{(0, y) : y > 0\}$ and $\{(t, 0) : t \geq 0\}$ by setting $\nu(0, y) = \sigma_0$ and $\nu(t, 0) = \sigma^*(t)$, where $\sigma^*(t) := \sigma_{i^*}(t)$ and $i^* = i^*(t)$ is such that $V_{i^*}(t) = \max_{i=1,\ldots,N} V_i(t)$. In particular, $\nu(0, 0) = \sigma_0$. Indeed, for every $\bar{y} > 0$ and every $\bar{t} \geq 0$,

$$\lim_{t \to 0} \nu(t, \bar{y}) = \sigma_0,$$
$$\lim_{y \to 0} \nu(\bar{t}, y) = \sigma^*(\bar{t}).$$

The function σ^* can in principle be discontinuous. However, we can easily make it a continuous function by assuming that $\sigma_1(t) \leq \sigma_2(t) \leq \cdots \leq \sigma_N(t)$ for each t, so that $\sigma^*(t) = \sigma_N(t)$ for each t. The continuity of the extension of ν on $\{(t, 0) : t \geq 0\}$ can then be straightforwardly proved.

Remark 3.2. *The above proposition provides us with the analytical expression for the diffusion coefficient in the SDE (16) such that the resulting equation has a unique strong solution whose marginal density is given by (17), i.e. (5) with* p^i*'s as in (14). Moreover, the square of the "local volatility"* $\nu(t, y)$ *can be viewed as a weighted average of the squared "basic volatilities"* $\sigma_1^2(t), \ldots, \sigma_N^2(t)$, *where the weights are all functions of the lognormal* *marginal densities (14). In particular, the "local volatility" $\nu(t,y)$ lies in the interval $[\tilde{\sigma},\hat{\sigma}]$.*[7] *In case $\sigma_1(t) \le \sigma_2(t) \le \cdots \le \sigma_N(t)$ for each t, we can actually prove, for each fixed t, the tighter inequalities*

$$\tilde{\sigma} \le \min_y \nu(t,y) = \sqrt{\frac{\sum_{i=1}^N \frac{\lambda_i \sigma_i^2(t)}{V_i(t)} e^{-\frac{1}{8}V_i^2(t)}}{\sum_{i=1}^N \frac{\lambda_i}{V_i(t)} e^{-\frac{1}{8}V_i^2(t)}}} \le \nu(t,y) \le \max_{i=1,\dots,N} \sigma_i(t) = \sigma^*(t) \le \hat{\sigma}.$$

Brigo and Mercurio (2001a) also remarked that, under deterministic interest rates, one can actually prove the existence of a unique risk-neutral measure, and hence forward measure. Indeed, let us assume that under the real-world measure Q_0, the asset price process evolves according to

$$dS_t = \mu_0 S_t dt + \nu(t,S_t) S_t dW_t^0,$$

where μ_0 is a real constant and W^0 is a Q_0-standard Brownian motion. Then the Radon-Nicodym derivative defining the change of measure from Q_0 to Q^T is expressed in terms of the "market price of risk"

$$\theta(t,S_t) = \frac{\mu_0 - \mu}{\nu(t,S_t)},$$

which is bounded due to (19). As a consequence, the Novikov condition, ensuring the feasibility of such a change of measure, is immediately fulfilled, and $dW_t = dW_t^0 + \theta(t,S_t)dt$.

As already pointed out, the pricing of European options under the asset-price model (16) with "local volatility" (15) is quite straightforward, see also Brigo and Mercurio (2001a).

Proposition 3.3. *Consider a European call option with maturity T, strike K and written on the asset S following the dynamics (16,15). Then, the option value at the initial time $t = 0$ is given by the following convex combination of Black-Scholes prices*

$$C(K,T) = P(0,T) \sum_{i=1}^N \lambda_i \left[S_0 e^{\mu T} \Phi\left(\frac{\ln \frac{S_0}{K} + \left(\mu + \frac{1}{2}\eta_i^2\right) T}{\eta_i \sqrt{T}} \right) - K \Phi\left(\frac{\ln \frac{S_0}{K} + \left(\mu - \frac{1}{2}\eta_i^2\right) T}{\eta_i \sqrt{T}} \right) \right], \tag{20}$$

where

$$\eta_i := \frac{V_i(T)}{\sqrt{T}} = \sqrt{\frac{\int_0^T \sigma_i^2(t)dt}{T}}. \tag{21}$$

Proof. We just have to apply (12) and notice that, in case of a call, $\mathcal{O}_i = P(t,T) \int_0^{+\infty} [(y - K)]^+ p_T^i(y) dy$ is nothing but the Black-Scholes call price corresponding to the volatility coefficient η_i. □

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[7] This property relates our model to that of Avellaneda et al. (1995) who considered a stochastic volatility evolving within a predefined band.

The option price (20) leads to smiles in the implied volatility structure. An example of the shape that can be reproduced in shown in Figure 1. Observe that the volatility implied by the option prices (20) has a minimum exactly at a strike equal to the forward asset price $S_0 e^{\mu T}$ (ATM forward strike). This property, which is formally proved in Brigo and Mercurio (2001a), makes the model suitable for recovering the smile-shaped volatility surfaces that are often observed in option markets. In fact, also skewed shapes can be retrieved, but with zero slope at the ATM-forward level.

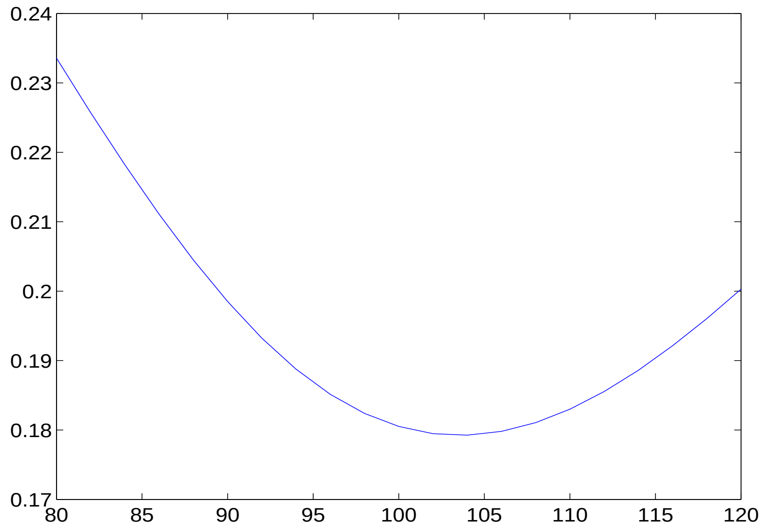

Figure 1: Implied volatility curve produced by the option price (20), where we set $\mu = 0.035$, $T = 1$, $N = 3$, $(\eta_1, \eta_2, \eta_3) = (0.6, 0.2, 0.1)$, $(\lambda_1, \lambda_2, \lambda_3) = (0.1, 0.3, 0.6)$ and $S_0 = 100$.

Given the above analytical tractability, we can easily derive an explicit approximation for the implied volatility as a function of the option strike price. More precisely, defining the moneyness m as the logarithm of the ratio between the forward asset price and the strike price, i.e.,

$$m := \ln \frac{S_0}{K} + \mu T,$$

we have the following.

Definition 3.4. *The Black-Scholes volatility that, for the given maturity T, is implied by the price (20) is the function $\hat{\sigma}(m)$ of the option moneyness that is implicitly defined by the equation*

$$\begin{aligned} P(0,T)S_0 e^{\mu T} & \left[\Phi\left(\frac{m + \frac{1}{2}\hat{\sigma}(m)^2 T}{\hat{\sigma}(m)\sqrt{T}} \right) - e^{-m} \Phi\left(\frac{m - \frac{1}{2}\hat{\sigma}(m)^2 T}{\hat{\sigma}(m)\sqrt{T}} \right) \right] \\ &= P(0,T) S_0 e^{\mu T} \sum_{i=1}^{N} \lambda_i \left[\Phi\left(\frac{m + \frac{1}{2}\eta_i^2 T}{\eta_i \sqrt{T}} \right) - e^{-m} \Phi\left(\frac{m - \frac{1}{2}\eta_i^2 T}{\eta_i \sqrt{T}} \right) \right]. \end{aligned} \tag{22}$$

Proposition 3.5. *The Black-Scholes volatility that is implied by the price (20) is given by*

$$\hat{\sigma}(m) = \hat{\sigma}(0) + \frac{1}{2\hat{\sigma}(0)T}\sum_{i=1}^{N}\lambda_i\left[\frac{\hat{\sigma}(0)}{\eta_i}e^{\frac{1}{8}(\hat{\sigma}(0)^2-\eta_i^2)T} - 1\right]m^2 + o(m^2), \tag{23}$$

where $\hat{\sigma}(0)$ is the ATM-forward implied volatility, which is explicitly given by

$$\hat{\sigma}(0) = \frac{2}{\sqrt{T}}\Phi^{-1}\left(\sum_{i=1}^{N}\lambda_i\Phi\left(\frac{1}{2}\eta_i\sqrt{T}\right)\right). \tag{24}$$

Proof. The definition (22) for $m = 0$ immediately leads to (24). As to the expansion, we just have to apply Dini's implicit function theorem and calculate the first and second derivatives in $m = 0$, obtaining

$$\frac{d\hat{\sigma}}{dm}(0) = \frac{\sum_{i=1}^{N}\lambda_i\Phi\left(-\frac{1}{2}\eta_i\sqrt{T}\right) - \Phi\left(-\frac{1}{2}\hat{\sigma}(0)\sqrt{T}\right)}{\frac{\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{1}{8}\hat{\sigma}(0)^2T}} = 0$$

$$\frac{d^2\hat{\sigma}}{dm^2}(0) = \frac{\sum_{i=1}^{N}\lambda_i\frac{e^{-\frac{1}{8}\eta_i^2T}}{\eta_i\sqrt{2\pi T}} - \frac{e^{-\frac{1}{8}\hat{\sigma}(0)^2T}}{\hat{\sigma}(0)\sqrt{2\pi T}}}{\frac{\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{1}{8}\hat{\sigma}(0)^2T}},$$

where the first derivative is zero due to (24). Straightforward algebra then leads to (23). □

The above model is quite appealing when pricing exotic derivatives. Notice, indeed, that having explicit dynamics implies that the asset-price paths can be simulated by discretizing the associated SDE with a numerical scheme. Hence we can use Monte Carlo procedures to price path-depending derivatives. Claims with early-exercise features can be priced with grids or lattices that can be constructed given the explicit form of the asset-price diffusion dynamics. However, the presence of a relative minimum at the ATM-forward level may be a severe drawback in case of highly skewed or asymmetric implied volatility curves. In such situations the following two models are in fact more suitable.

4 Lognormal-Mixtures with Different Means

We now consider the case where the densities p_t^i's are still lognormal but with different means. Precisely, we assume that the instrumental processes S_i's evolve, under Q^T, according to

$$dS_i(t) = \mu_i(t)S_i(t)dt + \sigma_i(t)S_i(t)\,dW_t, \quad i = 1,\ldots,N, \quad S_i(0) = S_0\ ,$$

where σ_i's are the same deterministic functions as in (13) satisfying the conditions of Proposition 3.1, and μ_i's are deterministic functions of time defined on $[0, T^*]$. The unconditional density of S_i at time t is thus given by

$$\begin{aligned} p_t^i(y) &= \frac{1}{yV_i(t)\sqrt{2\pi}} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{S_0} - M_i(t) + \tfrac{1}{2}V_i^2(t)\right]^2\right\}, \\ M_i(t) &:= \int_0^t \mu_i(u)du, \end{aligned} \tag{25}$$

with V_i defined as before. The functions μ_i's can not be defined arbitrarily, but must be chosen so that

$$\sum_{i=1}^{N} \lambda_i e^{M_i(t)} = e^{\mu t}, \quad \forall t > 0. \tag{26}$$

This is because, to be a proper Q^T-density, $p_t(y) = \sum_{i=1}^N \lambda_i p_t^i(y)$ must have a mean equal to $S_0 e^{\mu t}$.

Differentiating both sides of (26) with respect to t we get

$$\begin{aligned} &\sum_{i=1}^{N} \lambda_i \mu_i(t) e^{M_i(t)} = \mu e^{\mu t} = \mu \sum_{i=1}^{N} \lambda_i e^{M_i(t)}, \quad \forall t > 0, \\ &\Leftrightarrow \sum_{i=1}^{N} \lambda_i (\mu_i(t) - \mu) e^{M_i(t)} = 0, \quad \forall t > 0, \end{aligned}$$

which implies that some μ_i's must be larger and some smaller than (or equal to) μ.

The results of Section 2 can not be applied here because the instrumental processes S_i's no longer share the same drift rate μ. Nevertheless, we can apply a procedure similar to that of Brigo and Mercurio (2001a), see also Brigo and Mercurio (2002) for a more general treatment, and look for a diffusion coefficient $\psi(\cdot,\cdot)$ such that

$$dS(t) = \mu S(t)dt + \psi(t, S(t))S(t)\, dW_t \tag{27}$$

has a solution with marginal density $p_t(y) = \sum_{i=1}^N \lambda_i p_t^i(y)$. As before, we then use the Fokker-Planck equations for processes S and S_i's to find, after some manipulations and simplifications, that

$$\begin{aligned} \psi(t,y)^2 &= \frac{\sum_{i=1}^N \lambda_i \sigma_i(t)^2 p_t^i(y)}{\sum_{i=1}^M \lambda_i p_t^i(y)} + \frac{2\sum_{i=1}^N \lambda_i(\mu_i(t)-\mu)\int_y^{+\infty} x p_t^i(x)dx}{y^2 \sum_{i=1}^N \lambda_i p_t^i(y)} \\ &= \frac{\sum_{i=1}^N \lambda_i \sigma_i(t)^2 p_t^i(y)}{\sum_{i=1}^M \lambda_i p_t^i(y)} + \frac{2S_0 \sum_{i=1}^N \lambda_i(\mu_i(t)-\mu) e^{M_i(t)} \Phi\left(\frac{\ln\frac{S_0}{y} + M_i(t) + \frac{1}{2}V_i^2(t)}{V_i(t)}\right)}{y^2 \sum_{i=1}^N \lambda_i p_t^i(y)}. \end{aligned} \tag{28}$$

Notice that the first term in (28) coincides with $\nu^2(t,y)$ in (15) where the old densities (14) are now replaced with the new ones (25). Moreover, the integrals in the numerator of the second term in the RHS of (28) are quantities proportional to the Black-Scholes prices of asset or nothing options for the instrumental processes S_i.

The coefficient ψ is not necessarily well defined, since the second term in the RHS of (28) can become negative for some choices of the basic parameters, given that some μ_i's must be smaller than μ. However, it is possible to derive conditions under which (strict) positivity of $\psi(t,y)^2$ is granted. A set of sufficient conditions, not too restrictive from a practical viewpoint, is given in the following.

Lemma 4.1. *Assume that:*

i) there exists $n \in \{1,2,\dots,N\}$ *such that, for each* $t \in [0,T^*]$, $\mu_i(t) \geq \mu$ *for each* $i = 1,\dots,N$, $i \neq n$, *and* $\mu_n(t) \leq \mu$*;*

ii) the condition

$$\frac{V_i^2(t)}{2} - \frac{2V_i^2(t)}{\sigma_i^2(t)}(\mu_i(t)-\mu) > \frac{V_n^2(t)}{2} - \frac{2V_n^2(t)}{\sigma_n^2(t)}(\mu_n(t)-\mu) \tag{29}$$

is satisfied for each $t \in (0,T^*]$ *and for each* $i \neq n$,

then the function ψ^2 *in (28) is strictly positive on* $(0,T^*] \times (0,+\infty)$.

Proof. See Appendix A. □

If our model parameters and functions satisfy the assumptions of this lemma, it is then meaningful to deal with our candidate asset-price SDE (27). Further conditions ensuring existence and uniqueness of the solution of such an SDE are given in the following.

Proposition 4.2. *Let us assume that each* σ_i *is continuous and bounded from below by a positive constant, and that there exists an* $\varepsilon > 0$ *such that* $\sigma_i(t) = \sigma_0 > 0$, *for each* t *in* $[0,\varepsilon]$ *and* $i = 1,\dots,N$. *Let us further assume that each* μ_i *is continuous, that the no arbitrage condition (26) is satisfied, and that* $\mu_i(t) = \mu > 0$, *for each* t *in* $[0,\varepsilon]$ *and* $i = 1,\dots,N$. *Then, under the assumptions of Lemma 4.1, the SDE (27) has a unique strong solution whose marginal density is given by the mixture of lognormal densities (25).*

Proof. See Appendix B. □

The pricing of options, under dynamics (27), is again quite straightforward. In fact, the European call option price $\mathrm{C}(K,T)$, at time $t=0$, is again given by a convex combination of Black-Scholes prices, namely

$$\begin{aligned}&\mathrm{C}(K,T)\\ &= P(0,T)\sum_{i=1}^{N}\lambda_i\left[S_0 e^{M_i(T)}\Phi\!\left(\frac{\ln\frac{S_0}{K}+M_i(T)+\frac{1}{2}\eta_i^2 T}{\eta_i\sqrt{T}}\right) - K\Phi\!\left(\frac{\ln\frac{S_0}{K}+M_i(T)-\frac{1}{2}\eta_i^2 T}{\eta_i\sqrt{T}}\right)\right],\end{aligned} \tag{30}$$

where η_i is defined as in (21). Also this price leads to smiles in the implied volatility structure. However, the non-constant drift rates in the S_i-dynamics allows us to reproduce steeper and more skewed curves than in the zero-drifts case, with minimums that can be shifted far away from the ATM level.

5 The Case of Hyperbolic-Sine Processes

The third example we consider is another one lying in the class of dynamics (11). We in fact assume that the basic processes S_i evolve, under Q^T, according to hyperbolic-sine processes, i.e.[8]

$$S_i(t) = \beta_i(t) \sinh\left[\int_0^t \alpha_i(u)dW_u - L_i\right], \quad i = 1, \dots, N, \quad S_i(0) = S_0, \tag{31}$$

where α_i's are positive and deterministic functions of time, L_i's are negative constants, and β_i's are chosen so as to make S_i's drift rate equal to μ, namely

$$\beta_i(t) = \frac{S_0\, e^{\mu t - \frac{1}{2}\int_0^t \alpha_i^2(u)du}}{\sinh(-L_i)}.$$

The SDE followed by each S_i is thus given by

$$dS_i(t) = \mu S_i(t)dt + \alpha_i(t)\sqrt{\beta_i^2(t) + S_i^2(t)}\, dW_t, \quad i = 1, \dots, N.$$

Looking at this SDE's diffusion coefficient we immediately notice that it is roughly deterministic for small values of $S_i(t)$, whereas it is roughly proportional to $S_i(t)$ for large values of $S_i(t)$. Therefore in the former case, the dynamics are approximately of Gaussian type, whereas in the latter they are approximately of lognormal type. For further details on such a process we refer to Carr et al. (1999).[9]

The hyperbolic-sine process (31) shares all the analytical tractability of the classical geometric Brownian motion. This is intuitive, since (31) is basically the difference of two geometric Brownian motions (with perfectly negatively correlated logarithms).

Setting $A_i(t) := \sqrt{\int_0^t \alpha_i^2(u)du}$, the cumulative distribution function of process S_i at each time t is easily derived as follows:

$$\begin{aligned} Q^T\{S_i(t) \le y\} &= Q^T\left\{\int_0^t \alpha_i(u)dW_u \le L_i + \sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right\} \\ &= \Phi\left(\frac{L_i}{A_i(t)} + \frac{1}{A_i(t)}\sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right), \end{aligned}$$

so that the time-t marginal density of S_i is

$$p_t^i(y) = \frac{\exp\left\{-\frac{1}{2A_i^2(t)}\left[L_i + \sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right]^2\right\}}{A_i(t)\sqrt{2\pi}\sqrt{\beta_i^2(t) + y^2}}. \tag{32}$$

[8] We remind that $\sinh(x) = \frac{e^x - e^{-x}}{2}$, and that $\sinh^{-1}(x) = \ln(x + \sqrt{1+x^2})$.

[9] Carr et al. (1999) actually consider a process where negative values are absorbed into zero. Their process is slightly more complicated, though not losing in analytical tractability.

Moreover, through a straightforward integration, we obtain that the price of a European call with maturity T and strike K is

$$\mathrm{C}(T,K) = P(0,T)\left[\frac{S_0\, e^{\mu T}\Big(e^{-L_i}\Phi[\bar{y}_i(T)+A_i(T)] - e^{L_i}\Phi[\bar{y}_i(T)-A_i(T)]\Big)}{2\sinh(-L_i)} - K\Phi\big(\bar{y}_i(T)\big)\right], \tag{33}$$

where we set

$$\bar{y}_i(T) := -\frac{L_i}{A_i(T)} - \frac{1}{A_i(T)}\sinh^{-1}\left(\frac{K}{\beta_i(T)}\right).$$

The pricing function (33) leads to steeply decreasing patterns in the implied volatility curve. Therefore, we can hope that a mixture of densities (32) leads to steeper implied volatility skews than in the lognormal-mixture model. Indeed, this turns out to be the case.

The results in Section 2, and equation (11) in particular, immediately yield the following SDE for the asset price process under measure Q^T:

$$dS(t) = \mu S(t)dt + \chi(t, S(t))\, dW_t$$

$$\chi(t,y) = \sqrt{\frac{\sum_{i=1}^N \lambda_i \frac{\alpha_i^2(t)\sqrt{\beta_i(t)^2+y^2}}{A_i(t)} \exp\left\{-\frac{1}{2A_i^2(t)}\left[L_i + \sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right]^2\right\}}{\sum_{i=1}^N \frac{\lambda_i}{A_i(t)\sqrt{\beta_i(t)^2+y^2}} \exp\left\{-\frac{1}{2A_i^2(t)}\left[L_i + \sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right]^2\right\}}} \qquad (t,y) > (0,0). \tag{34}$$

As in the previous lognormal-mixtures cases, this equation must be handled with due care since the function χ is in general discontinuous in $(0, S_0)$ no matter what the value of $\chi(0, S(0))$ is. However, the existence and uniqueness of a solution of such SDE can again be proved under mild assumptions on the model coefficients. This is stated in the following.

Proposition 5.1. *Let us assume that each α_i is continuous and bounded from below by a positive constant, that there exists an $\varepsilon > 0$ such that $\alpha_i(t) = \alpha_0 > 0$, for each t in $[0, \varepsilon]$ and $i = 1, \ldots, N$, and that all L_i's are equal. Then, setting $\chi(0, S(0)) = \alpha_0$, we have that for $t \in [0, T]$, T a finite time horizon,*

$$C \le \chi^2(t,y) \le D(1+y^2). \tag{35}$$

Moreover, the SDE (34) admits a unique strong solution.

Proof. See Appendix C. □

The general treatment of Section 2 implies that the option price associated to (34) can be expressed in a closed form as follows.

Proposition 5.2. *Consider a European option with maturity T, strike K and written on the asset S following the dynamics (34). Then, the option value at the initial time $t = 0$ is given by the convex combination of prices (33), i.e.*

$$C(T, K) = P(0,T)\sum_{i=1}^{N} \lambda_i \left[\frac{S_0\, e^{\mu T}}{2\sinh(-L_i)} \Big(e^{-L_i}\Phi[\bar{y}_i(T) + A_i(T)] - e^{L_i}\Phi[\bar{y}_i(T) - A_i(T)] \Big) - K\Phi\big(\bar{y}_i(T)\big) \right]. \tag{36}$$

As anticipated, this option price leads to steep skews in the implied volatility curve. An example of the shape that can be reproduced is shown in Figure 2.

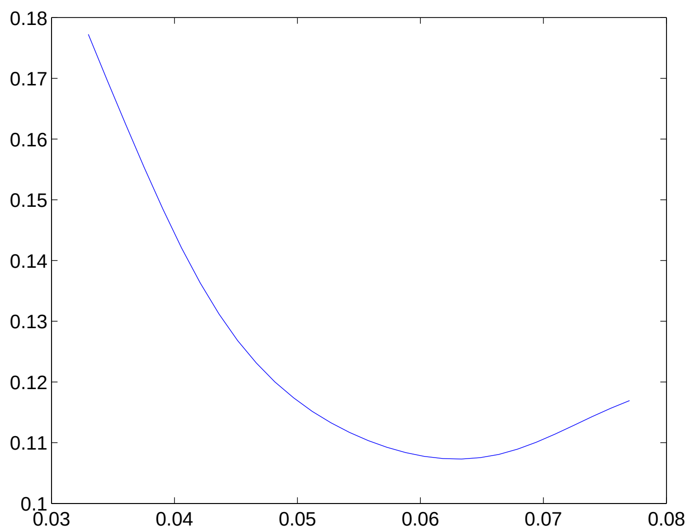

Figure 2: Implied volatility curve produced by the option price (36), where we set, $T = 1$, $N = 2$, $(A_1(1), A_2(1)) = (0.01, 0.04)$, $(L_1, L_2) = (-0.056, -0.408)$, $(\lambda_1, \lambda_2) = (0.1, 0.9)$, $\mu = 0$ and $S_0 = 0.055$.

6 Conclusions

We have reviewed the general class of asset-price models introduced by Brigo and Mercurio (2001a). This class is based on asset-price processes whose marginal density is given by the mixture of some suitably chosen densities. In particular, if the basic densities are associated to specific Q^T-asset-price dynamics that imply an analytical option price, so does their mixture.

We have then considered three fundamental cases. The first example is based on a mixture of lognormal densities with equal means and is particularly useful in case of smile-shaped implied volatilities. The second is still based on lognormal densities, but with possibly different means, whereas the third is built upon hyperbolic-sine basic processes. These last two models are instead advised for a calibration to skew-shaped (asymmetric) structures.

We can construct more general processes by applying affine transformations. For instance, in the case of lognormal densities with equal means, an alternative asset-price process, under Q^T, is defined, for any $\gamma \neq 0$, by

$$dS_t = \mu S_t dt + \sqrt{\frac{\sum_{i=1}^N \lambda_i \sigma_i^2(t) \frac{1}{V_i(t)} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln \frac{S_t - \gamma e^{\mu t}}{S_0 - \gamma} - \mu t + \frac{1}{2} V_i^2(t) \right]^2 \right\}}{\sum_{i=1}^N \lambda_i \frac{1}{V_i(t)} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln \frac{S_t - \gamma e^{\mu t}}{S_0 - \gamma} - \mu t + \frac{1}{2} V_i^2(t) \right]^2 \right\}}} (S_t - \gamma e^{\mu t}) dW_t,$$

which has a marginal density that is given by shifting a mixture of lognormal densities by the quantity $\gamma e^{\mu t}$ at each time t. The corresponding option prices lead to an implied volatility structure whose minimum point is shifted from the asset forward price. We are thus able to obtain slightly more flexible structures to better fit the market volatility data. Examples of this model fitting quality to real market data have been investigated by Brigo and Mercurio (2000) using option prices on the Italian stock index and by Brigo and Mercurio (2001b) using Euro caplet volatilities.

Finally, our models can be applied to all situations where the basic Black and Scholes paradigm is used, such as the equity market (Black and Scholes' formula), the FX market (Garman and Kohlhagen's formula) and the interest-rate cap market (Black's formula). In particular, our formulations can be used to introduce an analytically tractable forward Libor model for recovering the smile in the caps and floors markets, see also Brigo and Mercurio (2001c).

References

[1] Andersen, L. and Brotherton-Ratcliffe, R. (1997) The Equity Option Volatility Smile: An Implicit Finite-Difference Approach. *Journal of Computational Finance* 1, 5-38.

[2] Avellaneda, M., Friedman, C., Holmes, R. and Samperi D. (1997) Calibrating Volatility Surfaces via Relative-Entropy Minimization. Preprint. Courant Institute of Mathematical Sciences. New York University.

[3] Avellaneda, M., Levy, A., and Paras, A. (1995) Pricing and Hedging Derivative Securities in Markets with Uncertain Volatilities. *Applied Mathematical Finance* 2, 73-88.

[4] Bhupinder, B. (1998) Implied Risk-Neutral Probability Density Functions from Option Prices: A Central Bank Perspective. In *Forecasting Volatility in the Financial Markets*, 137-167. Edited by Knight, J., and Satchell, S. Butterworth Heinemann. Oxford.

[5] Bibby, B.M., and Sørensen M. (1997) A hyperbolic diffusion model for stock prices. *Finance and Stochastics* 1, 25-42.

[6] Black, F. and Scholes, M. (1973) The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* 81, 637-659.

[7] Bouchouev, I. (2000) Analytic Formulas for Pricing Options with the Volatility Smile. Preprint. Koch Petroleum Group.

[8] Brace, A., Goldys, B., Klebaner, F., and Womersley, R. (2001). Market Model of Stochastic Implied Volatility with application to the BGM Model. Working Paper S01-1, Department of Statistics, University of New South Wales, Sydney.

[9] Breeden, D.T. and Litzenberger, R.H. (1978) Prices of State-Contingent Claims Implicit in Option Prices. *Journal of Business* 51, 621-651.

[10] Brigo, D., and Mercurio, F. (2000). A mixed-up smile. *Risk* September, 123-126.

[11] Brigo, D., Mercurio, F. (2001a) Displaced and Mixture Diffusions for Analytically-Tractable Smile Models. In *Mathematical Finance - Bachelier Congress 2000*, Geman, H., Madan, D.B., Pliska, S.R., Vorst, A.C.F., eds. Springer Finance, Springer, Berlin, 151-174.

[12] Brigo, D., Mercurio, F. (2001b) Lognormal-Mixture Dynamics and Calibration to Market Volatility Smiles, International Journal of Theoretical & Applied Finance, forthcoming.

[13] Brigo, D., Mercurio, F. (2001c) *Interest Rate Models: Theory and Practice*. Springer Finance. Springer.

[14] Brigo, D., Mercurio, F. (2002) *On Stochastic Differential Equations with Marginal Laws Evolving According to Mixtures of Densities*. Internal Report. Banca IMI.

[15] Brigo, D., Mercurio, F. and Sartorelli, G. (2002) *Lognormal-Mixture Dynamics under Different Means*. Internal Report. Banca IMI.

[16] Britten-Jones, M. and Neuberger, A. (1999) Option Prices, Implied Price Processes and Stochastic Volatility. Preprint. London Business School.

[17] Brown, G. and Randall, C. (1999) If the Skew Fits. *Risk* April, 62-67.

[18] Carr, P., Tari, M. and Zariphopoulou T. (1999) Closed Form Option Valuation with Smiles. Preprint. NationsBanc Montgomery Securities.

[19] Cox, J. (1975) Notes on Option Pricing I: Constant Elasticity of Variance Diffusions. Working paper. Stanford University.

[20] Cox, J. and Ross S. (1976) The Valuation of Options for Alternative Stochastic Processes. *Journal of Financial Economics* 3, 145-166.

[21] Dempster, M.A.H. and Richard, D.G. (1999) Pricing Exotic American Options Fitting the Volatility Smile. Working Paper 17/99, Judge Institute of Management Studies, University of Cambridge.

[22] Derman, E. and Kani, I. (1994) Riding on a Smile. *Risk* February, 32-39.

[23] Derman, E. and Kani, I. (1998) Stochastic Implied Trees: Arbitrage Pricing with Stochastic Term and Strike Structure of Volatility. *International Journal of Theoretical and Applied Finance* 1, 61-110.

[24] Dupire, B. (1994) Pricing with a Smile. *Risk* January, 18-20.

[25] Dupire, B. (1997) Pricing and Hedging with Smiles. *Mathematics of Derivative Securities*, edited by M.A.H. Dempster and S.R. Pliska, Cambridge University Press, Cambridge, 103-111.

[26] Guo, C. (1998) Option Pricing with Heterogeneous Expectations. *The Financial Review* 33, 81-92.

[27] Heston, S. (1993) A Closed Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *Review of Financial Studies* 6, 327-343.

[28] Hull, J. and White, A. (1987) The Pricing of Options on Assets with Stochastic Volatilities. *Journal of Financial and Quantitative Analysis* 3, 281-300.

[29] Jackwerth, J.C. and Rubinstein, M. (1996) Recovering Probability Distributions from Option Prices. *Journal of Finance* 51, 1611-1631.

[30] Ledoit, O. and Santa-Clara, P. (1998) Relative Pricing of Options with Stochastic Volatility. Working paper, Anderson Graduate School of Management, University of California, Los Angeles.

[31] Li, A. (1998) A one Factor Volatility Smile Model with Closed Form Solutions for European Options. *European Financial Management*, 203-222.

[32] Malz, A.M. (1997) Option-Implied Probability Distributions and Currency Excess Returns. Preprint. Federal Reserve Bank of New York.

[33] Melick, W.R., and Thomas, C.P. (1997) Recovering an Asset's Implied PDF from Option Prices: An Application to Crude Oil During the Gulf Crisis. *Journal of Financial and Quantitative Analysis* 32, 91-115.

[34] Merton, R.C. (1976) Option Pricing When Underlying Stock Returns Are Discontinuous. Journal of Financial Economics, 3, 125-144.

[35] Pirkner, C.D., Weigend, A.S. and Zimmermann H. (1999) Extracting Risk-Neutral Densities from Option Prices Using Mixture Binomial Trees. Preprint. Stern School of Business.

[36] Prigent, J.L, Renault O. and Scaillet O. (2001) An Autoregressive Conditional Binomial Option Pricing Model. In *Mathematical Finance - Bachelier Congress 2000*, Geman, H., Madan, D.B., Pliska, S.R., Vorst, A.C.F., eds. Springer Finance, Springer, Berlin, 353- 373.

[37] Rebonato, R. (2001). The stochastic volatility Libor market model. *Risk* October, - .

[38] Ritchey, R.J. (1990) Call Option Valuation for Discrete Normal Mixtures. *Journal of Financial Research* 13, 285-296.

[39] Rogers, L.C.G. and Williams, D. (1996) Diffusions, Markov Processes and Martingales. Volume 2. John Wiley & Sons. Chichester.

[40] Rubinstein, M. (1994) Implied Binomial Trees. *Journal of Finance* 49, 771-818.

[41] Shimko, D. (1993) Bounds of Probability. *Risk* April, 33-37.

[42] Schönbucher, P. (1999) A Market Model of Stochastic Implied Volatility. *Philosophical Transactions of the Royal Society*, Series A, Vol. 357, No. 1758, pp. 2071-2092.

[43] Tompkins, R.G. (2000a) Stock Index Futures Markets: Stochastic Volatility Models and Smiles. Preprint. Department of Finance. Vienna University of Technology.

[44] Tompkins, R.G. (2000b) Fixed Income Futures Markets: Stochastic Volatility Models and Smiles. Preprint. Department of Finance. Vienna University of Technology.

Appendix A: Proof of Lemma 4.1

When $y \neq 0$, the function $\psi(t,y)^2$ in (28) is positive if and only if the function

$$y \to \sum_{i=1}^{N} \lambda_i \left[\sigma_i(t)^2 p_t^i(y) y^2 + 2(\mu_i(t) - \mu) \int_y^{+\infty} x p_t^i(x) dx \right] \tag{37}$$

is positive in $(0, +\infty)$ for each $t \in (0, T^*]$. We set

$$h_i(t,y) := \sigma_i(t)^2 p_t^i(y) y^2 + 2(\mu_i(t) - \mu) \int_y^{+\infty} x p_t^i(x) dx, \tag{38}$$

which, for each t and i, can be extended by continuity in $y = 0$ by setting

$$h_i(t,0) := \lim_{y \to 0} h_i(t,y) = 2(\mu_i(t) - \mu) S_0 e^{M_i(t)}.$$

Let us fix $t \in (0, T^*]$. As a function of y, h_i is strictly increasing up to the point

$$y_i := M_i(t) + \frac{V_i^2(t)}{2} - \frac{2V_i^2(t)}{\sigma_i^2(t)}(\mu_i(t) - \mu),$$

where it reaches a positive maximum, and then decreasing to 0, which is its limit for $y \to +\infty$.

Now, thanks to the no-arbitrage condition, $\sum_i \lambda_i h_i(t, 0) = 0$, and we also know that all h_i's are strictly increasing up to $y_* := \min_i y_i$, so that $\sum_i \lambda_i h_i$ must be strictly positive up to this point. Since $M_i(t) \geq M_n(t)$ for all $i \neq n$, (29) implies that $y_* = y_n$. Moreover, all h_i's, $i \neq n$, are strictly positive everywhere and h_n is strictly positive for $y \geq y_n$, so that $\sum_i \lambda_i h_i$ is strictly positive also for $y \geq y_n$.

Appendix B: Proof of Proposition 4.2

We first introduce the following.

Lemma 6.1. *The function ψ^2 in (28) is bounded on $[0, T] \times [0, +\infty)$ for any $T \in (0, T^*]$.*

Proof. Let $T > 0$ be fixed. We will first show that, for any $z > 0$, ψ^2 is bounded on $[0, T] \times [0, z]$, and then we will show that, for a suitable $\bar{z}$, ψ^2 is bounded on $[0, T] \times (\bar{z}, +\infty)$. By combining these results, and setting $z = \bar{z}$, we will finally obtain that ψ^2 is bounded on the whole domain $[0, T] \times [0, +\infty)$.

We define the following constants (which only depend on T):

$$V := \max_i \sup_{t\in[0,T]} V_i(t) \quad M := \max_i \sup_{t\in[0,T]} M_i(t) \quad m := \min_i \inf_{t\in[0,T]} M_i(t). \tag{39}$$

We denote by $\phi(t, y)$ the second term in (28). Since the first term in (28) is bounded, we need only show that $\phi(t, y)$ is bounded.

We first show that ϕ is continuous on $[0, T] \times [0, +\infty)$. We point out that ϕ is continuous on $[0, T] \times (0, +\infty)$ by its very definition, and it is equal to zero on the set $[0, T] \times \{0\}$. It is not hard to show that if $\lim_{y\to 0} \psi^2(t, y) = 0$ uniformly with respect to $t \in [0, T]$, then $\psi^2(t, y)$ is continuous on the points $(t, 0), t \in [0, T]$ too. To this end we could find an upper bound for the function ϕ which is uniform in $t \in [0, T]$ and converges to zero when $y \to 0$. However, as shown in Brigo et al. (2002), the limit is uniform in t also when there exists an upper bound for the ratio of the derivatives of the numerator and denominator in ϕ. In fact, for each $y < S_0 e^m$, we have

$$\left| \frac{2\sum_i \lambda_i(\mu - \mu_i(t))p_t^i(y)}{\sum_i \lambda_i p_t^i(y)\left(1 - \frac{\ln\frac{y}{S_0} - M_i(t) + \frac{1}{2}V_i^2(t)}{V_i^2(t)}\right)} \right| \leq 2 \sup_{t\in[0,T]} \left(\sum_i |\mu - \mu_i(t)|\right) \cdot \frac{V^2}{m - \ln(y/S_0)},$$

which implies boundedness on $[0, T] \times [0, z]$ for any $z > 0$.

The boundedness of ψ^2 on $[0,T] \times (\bar{z}, +\infty)$ is implied by the following bound which holds true for $y > S_0 e^{M+V^2/2}$, see Brigo et al. (2002),

$$\left| \frac{2\sum_i \lambda^i(\mu_i(t) - \mu) \int_y^{+\infty} x p_t^i(x) dx}{y^2 \sum_i \lambda^i p_t^i(z)} \right|$$
$$\leq 2(\sup_{t\in[0,T]} \sum_i |\mu_i(t) - \mu|) \int_0^{+\infty} \exp\left(-\frac{1}{2}\frac{u}{V^2}[u + 2\ln(y/S_0) - 2M - V^2] \right) du.$$

The limit of the RHS of this inequality for $y \to 0$ is 0 thanks to the Lebesgue dominated convergence theorem. This implies that $\lim_{y\to 0} \phi(t,y) = 0$ uniformly in $t \in [0,T]$, and hence there exists a $\bar{z}$ such that $|\phi(t,y)| \leq 1$ on $[0,T] \times [\bar{z}, +\infty)$. □

We can now move to the proof of Proposition 4.2. We apply the transformation $X_t = \ln(S_t)$ and study the equation

$$dX_t = \left(\mu - \frac{\psi(t, e^{X_t})^2}{2} \right) dt + \psi(t, e^{X_t}) dW_t. \tag{40}$$

To show existence and uniqueness of the solution to this equation we exploit Theorem 12.1, Section V.12, of Rogers and Williams (1996). We need to show that the drift and diffusion coefficients of this equation have linear growth with respect to y on domains like $[0,T] \times (-\infty, +\infty)$, $T > 0$, and are locally Lipschitz on domains like $[0,T] \times [-K, K]$, $T > 0$, $K > 0$.

The linear growth condition is satisfied due to the boundedness of both coefficients, which follows from the boundedness of $\psi(t, e^y)$ on those domains (see the lemma above). Moreover, the local Lipschitz condition is satisfied due to the fact that $\psi(t, e^y)^2$ is continuous, positive (see Lemma 4.1) and has a continuous derivative with respect to y on each set $[0,T] \times (-\infty, +\infty)$, $T > 0$. As a consequence, $\psi(t, e^y)^2$ is bounded from above and below by positive constants, and its y-partial derivative is bounded on each domain $[0,T] \times [-K, K]$, $K > 0$. Then, since $\frac{\partial}{\partial y}\psi(t, e^y) = \frac{1}{2\psi(t,e^y)} \frac{\partial}{\partial y}\psi(t, e^y)^2$, also $\psi(t, e^y)$ has a bounded y-partial derivative on $[0,T] \times [-K, K]$.

Appendix C: Proof of Proposition 5.1

Inequalities (35) immediately follow from the representation (9) and the assumptions on α_i's and L_i's.

The proof of the existence and uniqueness of the solution to the SDE (34), which is similar to that of Proposition 3.1, is again based on Theorem 12.1 in Section V.12 of Rogers and Williams (1996).

Given that the linear-growth condition is implied by (35), we just have to show that $\chi(t,y)$ is locally Lipschitz in the sense of this theorem. This is true since $\frac{\partial \chi^2}{\partial y}(t,y)$ is well defined and continuous for each $(t,y) \in [0,M] \times \mathbb{R}$, $M > 0$, and hence bounded on each compact set $[0,M] \times [-M, M]$. In fact, the same applies to $\frac{\partial \chi}{\partial y}(t,y) = \frac{1}{2\chi(t,y)} \frac{\partial \chi^2}{\partial y}(t,y)$ since $\chi(t,y)$ is bounded from below by a positive constant.