Lognormal-Mixture Dynamics under Different Means

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Abstract

We prove existence and uniqueness of the strong solution to the SDE whose drift rate is a given constant and whose diffusion coefficient is defined so as to imply a marginal density that is given by a mixture of lognormal densities. Such densities, which can have different means, must fulfill the no-arbitrage conditions that typically arise in mathematical finance when using the given SDE for modeling the price dynamics of some financial asset.

1 Definitions and notation

On a given filtered probability space, we consider the SDE

$$dS(t) = \mu S(t)dt + \psi(t, S(t))S(t)\, dW_t, \quad S(0) = S_0 > 0. \tag{1}$$

where W is standard Brownian motion, μ is a real number and ψ is the function, defined on $[0, +\infty) \times [0, +\infty) \to \mathbb{R}$, which, for any $(t, y) \in (0, +\infty) \times (0, +\infty)$, is given by $\psi(t, y) = \sqrt{\Psi(t, y)}$ where

$$\Psi(t, y) := \frac{\sum_{i=1}^{N} \lambda_i \sigma_i(t)^2 p_t^i(y)}{\sum_{i=1}^{M} \lambda_i p_t^i(y)} + \frac{2 \sum_{i=1}^{N} \lambda_i (\mu_i(t) - \mu) \int_y^{+\infty} x p_t^i(x) dx}{y^2 \sum_{i=1}^{N} \lambda_i p_t^i(y)}, \tag{2}$$

and, on the sets $\{0\} \times [0, +\infty)$ and $(0, +\infty) \times \{0\}$, is given by:

$$\psi(0, y) := \lim_{t \to 0} \sqrt{\Psi(t, y)} \quad \psi(t, 0) := \lim_{y \to 0} \sqrt{\Psi(t, y)}. \tag{3}$$

We assume that:

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- λ_i's are positive real numbers such that $\sum_{i=1}^{N} \lambda_i = 1$;
- σ_i's are continuous functions, all bounded below by a positive constant;
- $V_i(t)$'s are defined by: $V_i(t) := \sqrt{\int_0^t \sigma_i(s)^2 ds}$;
- μ_i's are continuous functions;
- $M_i(t)$'s are defined by: $M_i(t) := \int_0^t \mu_i(s) ds$;
- $p_t^i(y)$'s are lognormal density functions, i.e.

$$p_t^i(y) := \frac{1}{yV_i(t)} \exp\left[-\frac{1}{2} \left(\frac{\ln(y/S_0) - M_i(t) + V_i^2(t)/2}{V_i(t)} \right)^2 \right].$$

We adopt the following notation:

$$g_t^i(y) := \frac{\ln(y/S_0) - M_i(t) + \frac{V_i^2(t)}{2}}{V_i(t)}, \quad f(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

so that

$$\frac{d}{dy} g_t^i(y) = \frac{1}{yV_i(t)}, \quad \frac{d}{dx} f(x) = -xf(x).$$

Moreover,

$$p_t^i(y) = \frac{1}{yV_i(t)} f(g_t^i(y)),$$

$$\frac{d}{dy} p_t^i(y) = -\frac{1}{y} [1 + \frac{g_t^i(y)}{V_i(t)}] p_t^i(y).$$

We also assume that

$$\sum_{i=1}^{N} \lambda_i e^{M_i(t)} = e^{\mu t}, \quad \forall t > 0, \tag{4}$$

which can be interpreted as the typical no-arbitrage condition arising in mathematical finance, when the process S is the price of some financial asset and the initial probability measure is assumed to be the (unique) equivalent risk-neutral measure.

2 Good definition of ψ

The coefficient ψ is not necessarily well defined, since the second term in the RHS of (2) can become negative for some choices of the basic parameters due to (4), which implies (after differentiation) that some μ_i's must be smaller than μ. Moreover the limits in (3) may not exist. However, we can find sufficient conditions on these parameters which guarantee that function Ψ is strictly positive on its domain. An example of such conditions is given in the following.

Lemma 2.1. *Assume that:*

- *All μ_i's satisfy $\mu_i(t) \geq \mu$, $\forall t$ and for all i with one exception, say μ_n, and $\mu_n \leq \mu$, $\forall t$.*
- *Condition*
$$\frac{V_i^2}{2} - \frac{2V_i^2}{\sigma_i^2}(\mu_i - \mu) > \frac{V_n^2}{2} - \frac{2V_n^2}{\sigma_n^2}(\mu_n - \mu) \tag{5}$$
is satisfied for each $t \in (0, +\infty)$ and $\forall i \neq n$.

Then function (2) is strictly positive in $(0, +\infty) \times (0, +\infty)$.

Proof. When $y \neq 0$, function $\Psi(t, y)$ in (2) is positive if and only if function

$$\sum_{i=1}^{N} \lambda_i[\sigma_i(t)^2 p_t^i(y) y^2 + 2(\mu_i(t) - \mu) \int_y^{+\infty} x p_t^i(x) dx] \tag{6}$$

is positive. We set

$$h_i(t, y) := \sigma_i(t)^2 p_t^i(y) y^2 + 2(\mu_i(t) - \mu) \int_y^{+\infty} x p_t^i(x) dx \tag{7}$$

and it is easy to show that for any fixed t and i, $h_i(t, 0) = S_0 e^{M_i(t)}$ and, as a function of y, h_i is strictly increasing up to point $y_i := M_i(t) + \frac{V_i^2(t)}{2} - \frac{2V_i^2(t)}{\sigma_i^2(t)}(\mu_i(t) - \mu)$, where it reaches a positive maximum, and then it decreases to 0, which is its limit when $y \to +\infty$. Let then t be fixed. We know, thanks to the no-arbitrage condition, that $\sum_i \lambda_i h_i(t, 0) = 0$, and we know that all h_i's are strictly increasing up to $\min_i y_i$, so that $\sum_i \lambda_i h_i$ must be strictly positive up to point $\min_i y_i$. If condition (5) is satisfied, since $M_i(t) \geq M_n$ for all $i \neq n$, we have $\min_i y_i = y_n$. Since all h_i's, $i \neq n$, are strictly positive everywhere and h_n is strictly positive for $y \geq y_n$, then $\sum_i \lambda_i h_i$ is strictly positive for $y \geq y_n$. □

We now introduce the following assumptions, which guarantee existence of the limits in (3).

(H1) $\exists \varepsilon > 0 : \quad \sigma_i(t) = \sigma \quad \forall t \in [0, \varepsilon], \forall i,$

(H2) $\exists \varepsilon > 0 : \quad \mu_i(t) = \mu \quad \forall t \in [0, \varepsilon], \forall i.$

3 Boundedness of ψ

We want to show that ψ^2 is bounded on each domain like $[0, T] \times [0, +\infty)$, where T is a positive real number. We set:

$$V := \max_i \sup_{t \in [0,T]} V_i(t) \quad M := \max_i \sup_{t \in [0,T]} M_i(t) \quad m := (\min_i \inf_{t \in [0,T]} M_i(t)) \wedge 0 \tag{8}$$

We remark we have not taken the absolute value of $M_i(t)$ before taking the sup and the max or the inf and the min. M and m thus defined are the bounds we need.

The proof will proceed as follows: we split the domain $[0,T]\times[0,+\infty)$, into the sets $[0,T]\times[0,\bar{z}]$ and $[0,T]\times(\bar{z},+\infty)$ for a suitable $\bar{z}$, and we will show that ψ^2 is bounded on both sets, respectively.

In order to prove that ψ^2 is bounded on $[0,T]\times[0,\bar{z}]$, we start by pointing out that the first addendum in (2) is bounded. We then need only show that the second addendum in (2) is bounded. Let us call this function $\nu(t,y)$.[1] In order that the definition of ν be coherent with that of ψ^2, we set $\nu(t,0)=0\ \forall t$. We now show that ν thus defined is a continuous function on $[0,T]\times[0,\bar{z}]$, and hence bounded on such a domain. To this end, we will exploit Lemmas 3.1 and 3.2: first we will find an upper bound for the function (9) below, then we will apply Lemma 3.1 to show that $\lim_{x\to 0}\nu(t,x)=0$ uniformly with respect to $t\in[0,T]$, and finally we will apply Lemma 3.2 to see that ν is continuous as stated above.

$$\left|\frac{2\sum_i \lambda_i(\mu-\mu_i(t))p_t^i(z)}{\sum_i \lambda_i p_t^i(z)(1-\frac{g_t^i(z)}{V_i(t)})}\right| \leq \tag{9}$$

$$\leq 2\sum_i \frac{\lambda_i|\mu-\mu_i(t)|p_t^i(z)}{\lambda_i p_t^i(z)(1-\frac{g_t^i(z)}{V_i(t)})} =\leq 2\sum_i \frac{|\mu-\mu_i(t)|}{(1-\frac{g_t^i(z)}{V_i(t)})}.$$

The above inequalities hold true as long as $(1-\frac{g_t^i(z)}{V_i})\geq 0$, which is satisfied if $z\leq S_0e^m$. Under this constraint, we also obtain the following upper bound:

$$\frac{1}{1-\frac{g_t^i(z)}{V_i(t)}} \leq \frac{V^2}{m-\ln(z/S_0)},$$

so that, finally,

$$\left|\frac{2\sum_i \lambda_i(\mu-\mu_i(t))p_t^i(z)}{\sum_i \lambda_i p_t^i(z)(1-\frac{g_t^i(z)}{V_i(t)})}\right| \leq 2\sup_{t\in[0,T]}\left(\sum_i|\mu-\mu_i(t)|\right)\cdot\frac{V^2}{m-\ln(z/S_0)}$$

for all $t\in[0,T]$ and for all $z\in(0,S_0e^m]$. As mentioned above, we now apply the following lemma to get that $\lim_{y\to 0}\nu(t,y)=0$ uniformly in $t\in[0,T]$.

Lemma 3.1. *Let $f,g:[0,T]\times(0,+\infty)\to\mathbb{R}$ be continuous functions such that:*

$$\lim_{x\to 0} f(t,x)=0, \quad \lim_{x\to 0} g(t,x)=0, \quad \forall t\in[0,T].$$

Assume that both functions admit x-partial derivatives on their whole domain, that $g(t,x)\neq 0\ \forall t,x$ on its domain, and that there exists a function $F:(0,K)\to\mathbb{R}$ such that

$$\left|\frac{f'_x(t,x)}{g'_x(t,x)}\right| \leq F(x), \quad \forall t\in[0,T], \quad \forall x\in[0,K], \quad \textit{and} \quad \lim_{x\to 0}F(x)=0,$$

[1] Notice that ν is not well defined on $[0,T]\times\{0\}$.

for a suitable K. Then

$$\lim_{x\to 0}\frac{f(t,x)}{g(t,x)}=0$$

uniformly for $t\in[0,T]$.

Proof. Let $\varepsilon>0$ be fixed. Take $\delta>0$ such that $|x|<\delta \implies F(x)<\varepsilon/2$. Now take an arbitrary $r<\delta$. Then, for any $0<x<r$ and for all t, by a Cauchy (mean value) theorem, there exists y depending on t, x, r, such that:

$$y(t,x,r)\in(x,r),\quad \text{and}\quad \left|\frac{f(t,x)-f(t,r)}{g(t,x)-g(t,r)}\right|=\left|\frac{f'_x(t,y(t,x,r))}{g'_x(t,y(t,x,r))}\right|\le F(y(t,x,r))<\frac{\varepsilon}{2}$$

so that for any $0<x<r<\delta$, we have

$$\left|\frac{f(t,x)-f(t,r)}{g(t,x)-g(t,r)}\right|<\frac{\varepsilon}{2}.$$

If we take the limit for $x\to 0$, we get

$$\left|\frac{f(t,r)}{g(t,r)}\right|\le\frac{\varepsilon}{2}<\varepsilon,$$

which holds true for any $r\in(0,\delta)$. Since δ does not depend on t, not only do we get that $\lim_{x\to 0}\frac{f(t,x)}{g(t,x)}=0$, but also this limit is uniform with respect to $t\in[0,T]$. □

Now we can apply the following lemma to obtain that ν is continuous on $[0,T]\times[0,\bar{z}]$ for any $\bar{z}$.

Lemma 3.2. *Let $f:[0,T]\times(0,+\infty)\to\mathbb{R}$ be a continuous real function. Assume that*

$$\lim_{x\to 0}f(t,x)=0,\quad \forall t\in[0,T] \tag{10}$$

uniformly for $t\in[0,T]$, i.e. $\forall\varepsilon>0\ \exists\delta>0$ s.t. $|x|<\delta \implies |f(t,x)|<\varepsilon\ \forall t\in[0,T]$. Then, if we set $f(t,0):=0$, for all t, the function thus defined, $f:[0,T]\times[0,+\infty)\to\mathbb{R}$, is continuous.

Proof. We only have to show that

$$\lim_{(s,x)\to(t,0)}f(s,x)=0,\quad \forall t\in[0,T],$$

but this is immediate, since $\forall\varepsilon>0$, we have a $\delta>0$, such that $\forall(s,y)\in[0,T]\times[0,\delta)$, $|f(s,y)|<\varepsilon$; and $[0,T]\times[0,\delta)$ is a neighborhood of $(t,0)$ in $[0,T]\times[0,+\infty)$ with respect to its natural topology. □

We have thus proved that ν is continuous on $[0,T]\times[0,\bar{z}]$, for any $\bar{z}$. Weierstrass' theorem lets us conclude that ν is also bounded on any such domain.

In order to show there exists a $\bar{z}$ such that ψ^2 is also bounded on the set $[0,T]\times(\bar{z},+\infty)$, consider the two addenda in (2): we know that the first addendum is bounded on the set $[0,T]\times[0,+\infty)$, and hence we only need to show there exists a suitable (big enough) $\bar{z}$, such that the second addendum in (2) is bounded on the set $[0,T]\times(\bar{z},+\infty)$. Consider the following inequalities:

$$\left|\frac{2\sum_i\lambda^i(\mu_i(t)-\mu)\int_z^{+\infty}yp_t^i(y)dy}{z^2\sum_i\lambda^ip_t^i(z)}\right|\leq\frac{2\sum_i\lambda^i|\mu_i(t)-\mu|\int_z^{+\infty}yp_t^i(y)dy}{z^2\sum_i\lambda^ip_t^i(z)}\leq$$

$$\leq 2\sum_i\frac{\lambda^i|\mu_i(t)-\mu|\int_z^{+\infty}yp_t^i(y)dy}{z^2\lambda^ip_t^i(z)}=2\sum_i|\mu_i(t)-\mu|\frac{\int_z^{+\infty}yp_t^i(y)dy}{z^2p_t^i(z)}.$$

We have:

$$\frac{\int_z^{+\infty}yp_t^i(y)dy}{z^2p_t^i(z)}=\frac{\int_z^{+\infty}f(g_t^i(y))dy}{zf(g_t^i(z))}=\int_z^{+\infty}\frac{1}{z}\exp(-\frac{1}{2}[g_t^i(y)^2-g_t^i(z)^2])dy.$$

We apply the change of variable $u=\ln(y)-\ln(z)$ in the last integral and get:

$$\int_0^{+\infty}\frac{1}{z}\exp(-\frac{1}{2}[(\frac{u}{V_i}+g_t^i(z))^2-g_t^i(z)^2])ze^udu= \tag{11}$$

$$=\int_0^{+\infty}\exp(-\frac{1}{2}\frac{u}{V_i^2}[u+2\ln(z/S_0)-2M_i(t)-V_i^2(t)])du$$

Now we observe that for any $z>S_0e^{M+V^2/2}$, $2\ln(z/S_0)-2M_i(t)-V_i^2(t)$ is positive, and hence, for any such a z, we have that

$$\frac{u}{V_i^2(t)}[u+2\ln(z/S_0)-2M_i(t)-V_i^2(t)]\geq\frac{u}{V^2}[u+2\ln(z/S_0)-2M_i(t)-V_i^2(t)]$$

Moreover:

$$\frac{u}{V^2}[u+2\ln(z/S_0)-2M_i(t)-V_i^2(t)]\geq\frac{u}{V^2}[u+2\ln(z/S_0)-2M-V^2],$$

and we bound the last integral above with the function:

$$\int_0^{+\infty}\exp(-\frac{1}{2}\frac{u}{V^2}[u+2\ln(z/S_0)-2M-V^2])du.$$

We remark that while this upper bound holds true only for $z>S_0e^{M+V^2/2}$, $S_0e^{M+V^2/2}$ does not depend on t or i and the bound is the same for every i. We thus have:

$$\left|\frac{2\sum_i\lambda^i(\mu_i(t)-\mu)\int_z^{+\infty}yp_t^i(y)dy}{z^2\sum_i\lambda^ip_t^i(z)}\right|\leq$$

$$\leq 2 \sum_i |\mu_i(t) - \mu| \int_0^{+\infty} \exp(-\frac{1}{2}\frac{u}{V^2}[u + 2\ln(z/S_0) - 2M - V^2])du \leq$$

$$\leq 2(\sup_{t\in[0,T]} \sum_i |\mu_i(t) - \mu|) \int_0^{+\infty} \exp(-\frac{1}{2}\frac{u}{V^2}[u + 2\ln(z/S_0) - 2M - V^2])du.$$

The sup appearing in this expression is a real number (depending on T) since all μ_i's are continuous functions. We recall that the inequality holds for $z > S_0 e^{M+V^2/2}$, and we point out that

$$\lim_{z\to+\infty} \int_0^{+\infty} \exp(-\frac{1}{2}\frac{u}{V^2}[u + 2\ln(z/S_0) - 2M - V^2])du = 0$$

thanks to Lebesgue's dominated convergence theorem. We only have to find a $\bar{z}$, bigger than $S_0 e^{M+V^2/2}$, such that for any $z > \bar{z}$

$$\left| \int_0^{+\infty} \exp(-\frac{1}{2}\frac{u}{V^2}[u + 2\ln(z/S_0) - 2M - V^2])du \right| \leq 1,$$

so that, finally:

$$\left| \frac{2\sum_i \lambda^i(\mu_i(t) - \mu)\int_z^{+\infty} y p_t^i(y)dy}{z^2 \sum_i \lambda^i p_t^i(z)} \right| \leq$$

$$\leq 2(\sup_{t\in[0,T]} \sum_i |\mu_i(t) - \mu|),$$

for any $(t, y) \in [0, T] \times (\bar{z}, +\infty)$.

4 Existence and Uniqueness of the solution to the SDE

Proposition 4.1. *Let us assume that each σ_i is continuous and bounded from below by a positive constant, and that there exists an $\varepsilon > 0$ such that $\sigma_i(t) = \sigma_0 > 0$, for each t in $[0, \varepsilon]$ and $i = 1, \ldots, N$. Let us further assume that each μ_i is continuous, that the no arbitrage condition (4) is satisfied, and that $\mu_i(t) = \mu > 0$, for each t in $[0, \varepsilon]$ and $i = 1, \ldots, N$. Finally, assume the hypotheses of Lemma 2.1 are satisfied. Then, the SDE*

$$dS_t = \mu S_t dt + \psi(t, S_t) S_t dW_t \tag{12}$$

has a unique strong solution whose marginal density is given by the mixture of lognormals

$$p_t(y) = \sum_{i=1}^N \lambda_i \frac{1}{y V_i(t)\sqrt{2\pi}} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln\frac{y}{S_0} - M_i(t) + \tfrac{1}{2}V_i^2(t) \right]^2 \right\}. \tag{13}$$

Proof. Instead of directly studying equation (12) we apply the transformation $X_t = \ln(S_t)$ and study equation

$$dX_t = (\mu - \frac{\psi(t, e^{X_t})^2}{2})dt + \psi(t, e^{X_t})dW_t. \tag{14}$$

To show existence and uniqueness of this equation we exploit Theorem 12.1, Section V.12, of Rogers and Williams (1996). We need to show that the coefficients of this equation have linear growth with respect to y on domains like $[0, T] \times (-\infty, +\infty)$, $T > 0$, and are locally Lipschitz on domains like $[0, T] \times [-K, K]$, $T > 0$, $K > 0$. The linear growth condition is satisfied due to the boundedness of both coefficients, which follows from boundedness of function $\psi(t, e^y)$ on those domains (see Section 3). The local Lipschitz condition is satisfied due to the fact that $\psi(t, e^y)^2$ is continuous, positive (see Lemma 2.1) and has a continuous derivative with respect to y on each set like $[0, T] \times (-\infty, +\infty)$. Hence, it is bounded, bounded away from zero and its y-partial derivative is bounded on each domain like $[0, T] \times [-K, K]$. Then, thanks to the equality $\frac{\partial}{\partial y}\psi(t, e^y) = \frac{1}{2\psi(t,e^y)}\frac{\partial}{\partial y}\psi(t, e^y)^2$, also $\psi(t, e^y)$ has a bounded y-partial derivative on each such a domain.

We finally remark that the hypotheses that $\sigma_i(t) = \sigma_0$ and $\mu_i = \mu$ $\forall t \in [0, \varepsilon)$ $\forall i$ are fundamental in proving continuity. □