PRICING THE SMILE IN A FORWARD LIBOR MARKET MODEL

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Motivation of the Paper

- The LIBOR market models are the most popular interest rate models.
- Their success is due to:
  - Capability of exactly recovering Black's formulas for caps.
  - Efficient analytical approximations of swaption volatilities.
  - Possibility of a joint calibration to the caps and swaptions markets.
- However, there are a number of open issues:
  - Choice of a suitable instantaneous covariance structure of forward rates.
  - Instantaneous correlations are inputs or outputs?
  - Pricing the smile.

Basic Definitions and Notation

- Let $t = 0$ be the current time and $\mathcal{T} = \{T_0, \ldots, T_M\}$ be a set of times from which expiry-maturity pairs (T_{k-1}, T_k) for a family of spanning forward rates are taken.

- Denote by $\{\tau_0, \ldots, \tau_M\}$ the corresponding year fractions: τ_k is the year fraction associated with the pair (T_{k-1}, T_k), $k > 0$, and τ_0 is the year fraction from today to T_0 (we set $T_{-1} := 0$).

- Denote by $P(t, T)$ the price at time t of the zero-coupon bond with maturity T.

- Denote by Q^k the T_k-*forward measure*, i.e. the probability measure associated with the numeraire $P(\cdot, T_k)$.

The Forward LIBOR Rate

- The forward LIBOR rate at time t with expiry T_{k-1} and maturity T_k is defined by

$$\boxed{F_k(t) =: F(t; T_{k-1}, T_k) = \frac{P(t, T_{k-1}) - P(t, T_k)}{\tau_k P(t, T_k)}}$$

  and is "alive" up to time T_{k-1}, $k = 1, \ldots, M$.

- Note that $F_k(t)P(t, T_k)$ is the price of a tradable asset, so that

$$\frac{F_k(t)P(t, T_k)}{P(t, T_k)} = F_k(t) \text{ is a martingale under } Q^k.$$

The Lognormal Forward LIBOR Model (LFM)

- Developed by Miltersen, Sandmann and Sondermann (1997), and Brace, Gatarek and Musiela (1997).

- They assumed the following driftless dynamics for F_k under Q^k:

$$dF_k(t) = \sigma_k(t) F_k(t)\, dZ_k(t), \quad t \leq T_{k-1},$$

where Z_k is the k-th component of an M-dimensional Q^k-Brownian motion Z with instantaneous correlation matrix $\rho = (\rho_{i,j})_{i,j=1,\dots,M}$, i.e.

$$dZ_i(t)\, dZ_j(t) = \rho_{i,j}\, dt,$$

and where the instantaneous volatility $\sigma_k(t)$ is deterministic.

LFM Dynamics under Different Numeraires

Proposition. *The LFM dynamics of F_k under the forward-adjusted measure Q^i in the two case cases $i < k$ and $i > k$ are, respectively*

$$i < k,\ dF_k(t) = \sigma_k(t)F_k(t)\sum_{j=i+1}^{k}\frac{\rho_{k,j}\,\tau_j\,\sigma_j(t)\,F_j(t)}{1+\tau_jF_j(t)}\,dt + \sigma_k(t)F_k(t)\,dZ_k(t)$$

$$i > k,\ dF_k(t) = -\sigma_k(t)F_k(t)\sum_{j=k+1}^{i}\frac{\rho_{k,j}\,\tau_j\,\sigma_j(t)\,F_j(t)}{1+\tau_jF_j(t)}\,dt + \sigma_k(t)F_k(t)\,dZ_k(t)$$

where $Z = Z^i$ is a standard Brownian motion under Q^i, and $t \le T_i \wedge T_{k-1}$.

- **N.B.** Each of the above equations admits a unique strong solution.

The Pricing of Caplets under the LFM

Definition. *The T_{k-1}-caplet is a contract whose payoff at time T_k is*

$$\tau_k[L(T_{k-1}, T_k) - K]^+ = \tau_k[F_k(T_{k-1}) - K]^+$$

Proposition. *The LFM price of the T_{k-1}-caplet coincides with that given by the corresponding Black caplet formula. In fact, at $t = 0$,*

$$\begin{aligned}
\textbf{Cpl}(T_{k-1}, T_k, K) &= \textbf{Cpl}^{\textit{Black}}(T_{k-1}, T_k, K, v_k) = P(0, T_k)\tau_k \mathsf{Bl}(K, F_k(0), v_k) \\
\mathsf{Bl}(K, F_k(0), v_k) &= E^{Q^k}(F_k(T_{k-1}) - K)^+ \\
&= F_k(0)\Phi(d_1(K, F_k(0), v_k)) - K\Phi(d_2(K, F_k(0), v_k)) \\
d_{1,2}(K, F, v) &= \frac{\ln(F/K) \pm v^2/2}{v}, \qquad v_k^2 = \int_0^{T_{k-1}} \sigma_k(t)^2 \, dt
\end{aligned}$$

Calibration to Caps Data

Definition. *A cap is a portfolio of caplets. Precisely, the* $\mathcal{T}$*-cap is a contract paying* $\tau_k[F_k(T_{k-1}) - K]^+$ *at each time* T_k.

- The calibration of the LFM to at-the-money caps is exact and automatic.
- But what about away-from-the-money caps (or even at-the-money caplets)?
- Empirical fact: $\mathcal{T}$-caps with different strikes are priced (quoted) with different implied volatilities.
- Therefore, there is the need for alternative forward LIBOR dynamics to price the market cap-volatility smile.

Main References

- Andersen and Andreasen (2000): CEV process.
- Balland and Hughston (2000): exact calibration to market cap vols.
- Rebonato (2001), Wu and Zhang (2002): stochastic volatility.
- Brace, Goldys, Klebaner and Womersley (2001): market model of implied volatility.
- Our works on density-mixtures models:
  - B. & M. (2000), Risk.
  - B. & M. (2001), Mathematical Finance, Bachelier Congress.
  - B. & M. (2001), *Interest Rate Models: Theory and Practice*, Springer.
  - B. & M. (2002), International Journal of Theo. & App. Finance.

The Lognormal-Mixture Forward Model (LMFM)

Proposition. *Let $\sigma_i(t)$, $i = 1, \ldots, N$, be deterministic, continuous and bounded from below by a positive constant. Assume there exists $\varepsilon > 0$ such that $\sigma_i(t) = \sigma_0 > 0$, for each t in $[0, \varepsilon]$ and $i = 1, \ldots, N$. If we set*

$$p_t^i(y) = \frac{1}{yV_i(t)\sqrt{2\pi}} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln \frac{y}{F_k(0)} + \tfrac{1}{2}V_i^2(t) \right]^2 \right\},$$

$$V_i(t) := \sqrt{\int_0^t \sigma_i^2(u)du}, \quad \nu(t, y) := \sqrt{\frac{\sum_{i=1}^N \lambda_i \sigma_i^2(t) p_t^i(y)}{\sum_{i=1}^N \lambda_i p_t^i(y)}},$$

then, under Q^k,

$$dF_k(t) = \nu(t, F_k(t))F_k(t)\, dZ_k(t)$$

has a unique strong solution with marginal density $p_t(y) = \sum_{i=1}^N \lambda_i p_t^i(y)$.

LMFM Calibration to Caps Price

- The LMFM is analytically tractable: a LMFM cap price is simply the mixture of Black's caps prices.

- The LMFM is easy to calibrate to caps data. It is more suitable for caplet volatilities with a minimum around the related forward rate.

- The LMFM can be extended by shifting its dynamics,

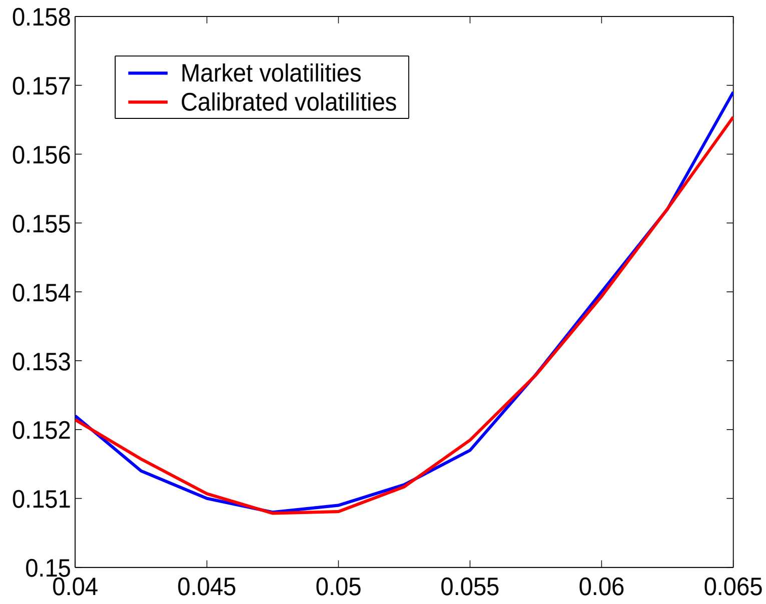
$$F_k(t) = \alpha + \bar{F}_k(t),$$

  to calibrate more asymmetric structures.

- The LMFM can also be extended to the case of lognormal densities with different means.

An Example of LMFM Calibration to Caplet Volatilities

- Data: Two-year Euro caplet volatilities as of November 14th, 2000 (LIBOR resetting at 1.5 years).
- We set $N = 2$ and minimize the squared percentage difference between model and market (mid) prices.



Hyperbolic-Sine Processes

- Consider N basic processes G_i, which evolve under Q^k according to

$$G_i(t) = \beta_i(t) \sinh\left[\int_0^t \alpha_i(u)\, dW_u - L_i\right], \quad i = 1, \dots, N, \quad G_i(0) = F_k(0),$$

  where α_i's are positive and deterministic functions of time, L_i's are negative constants, and

$$\beta_i(t) = \frac{F_k(0)\, e^{-\frac{1}{2}\int_0^t \alpha_i^2(u)du}}{\sinh(-L_i)}.$$

- The SDE followed by each G_i is given by

$$dG_i(t) = \alpha_i(t)\sqrt{\beta_i^2(t) + G_i^2(t)}\, dZ_k(t), \quad i = 1, \dots, N.$$

Hyperbolic-Sine Processes (cont'd)

- The time-t marginal density of G_i is

$$p_t^i(y) = \frac{\exp\left\{-\frac{1}{2A_i^2(t)}\left[L_i + \sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right]^2\right\}}{A_i(t)\sqrt{2\pi}\sqrt{\beta_i^2(t)+y^2}}, \quad A_i(t) := \sqrt{\int_0^t \alpha_i^2(u)du}$$

- The price of a European call with maturity T and strike K is

$$\mathsf{C}(T,K) = P(0,T)\frac{F_k(0)\,e^{\mu T}\left(e^{-L_i}\Phi\left[\bar{y}_i^T + A_i(T)\right] - e^{L_i}\Phi\left[\bar{y}_i^T - A_i(T)\right]\right)}{2\sinh(-L_i)}$$

$$\left. - P(0,T)K\Phi\left(\bar{y}_i^T\right)\right], \quad \bar{y}_i^T := -\frac{L_i}{A_i(T)} - \frac{1}{A_i(T)}\sinh^{-1}\left(\frac{K}{\beta_i(T)}\right)$$

The Hyperbolic-Sine-Density-Mixture Model (HSDMM)

Proposition. *Assume each α_i is continuous and bounded from below by a positive constant, that there exists an $\varepsilon > 0$ such that $\alpha_i(t) = \alpha_0 > 0$, for each t in $[0, \varepsilon]$ and $i = 1, \ldots, N$, and that all L_i's are equal. Then, setting*

$$\chi(t,y) := \sqrt{\frac{\sum_{i=1}^{N} \lambda_i \frac{\alpha_i^2(t)\sqrt{\beta_i(t)^2+y^2}}{A_i(t)} \exp\left\{-\frac{1}{2A_i^2(t)}\left[L_i + \sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right]^2\right\}}{\sum_{i=1}^{N} \frac{\lambda_i}{A_i(t)\sqrt{\beta_i(t)^2+y^2}} \exp\left\{-\frac{1}{2A_i^2(t)}\left[L_i + \sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right]^2\right\}}}$$

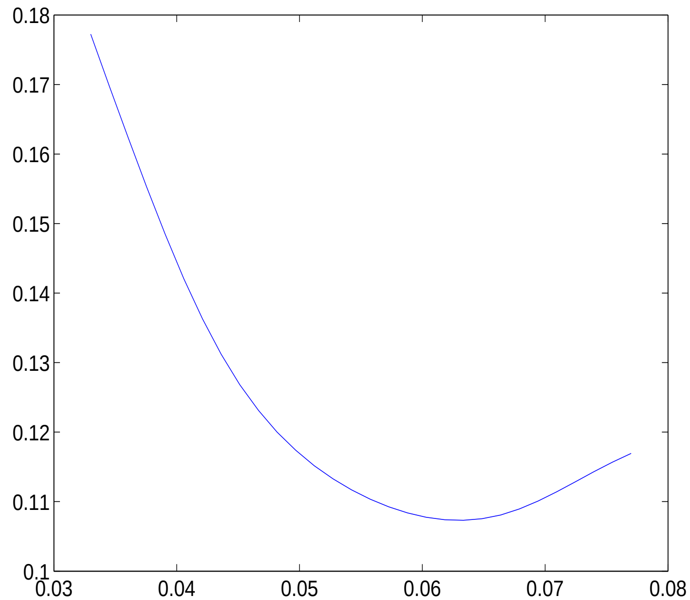
with $\chi(0, F_k(0)) = \alpha_0$, under Q^k the SDE

$$dF_k(t) = \chi(t, F_k(t))\, dZ_k(t)$$

admits a unique strong solution with marginal density $p_t(y) = \sum_{i=1}^{N} \lambda_i p_t^i(y)$.

Caplet Volatility Curves Implied by the HSDMM

- The HSDMM is analytically tractable: a caplet price is simply the mixture of the caplet prices associated to the basic hyperbolic-sine processes.
- The implied volatility curves are usually skew-shaped:



A New General Class

Assume that F_k can be expressed as

$$F_k(t) = h(t, W_t) = h_t(W_t) \text{ for each } 0 \leq t \leq T_{k-1},$$

where W is a standard Brownian motion and the function h satisfies:

- h belongs to $C^{1,2}(\mathcal{D})$, with $\mathcal{D} := [0, T_{k-1}] \times I\!R$;
- $h_t(w) > 0$ for each $(t, w) \in \mathcal{D}$;
- for each $t > 0$, $\lim_{w \to -\infty} h_t(w) = 0$ and $dh_t(w)/dw > 0$ (h_t is invertible and its inverse h_t^{-1} is differentiable);
- $E^k\{h_{T_{k-1}}(W_{T_{k-1}})\} < +\infty$ and $E_t^k\{h_{T_{k-1}}(W_{T_{k-1}})\} = h_t(W_t)$, $\forall t \leq T_{k-1}$.

Properties of the New Class

The associated SDE. By Ito's lemma:

$$dF_k(t) = \left[\frac{\partial h}{\partial t}(t, W_t) + \frac{1}{2}\frac{\partial^2 h}{\partial w^2}(t, W_t)\right] dt + \frac{\partial h}{\partial w}(t, W_t)\, dW_t$$
$$= \frac{\partial h}{\partial w}\left(t, h_t^{-1}(F_k(t))\right) dW_t = \sigma(t, F_k(t)) F_k(t)\, dW_t$$

Marginal density. Denoting by p_t the marginal density of $F_k(t)$:

$$Q^k\{F_k(t) \le x\} = Q^k\{h_t(W_t) \le x\} = Q^k\{W_t \le h_t^{-1}(x)\} = \Phi\left(\frac{h_t^{-1}(x)}{\sqrt{t}}\right)$$

$$p_t(x) = \frac{d}{dx} Q^k\{F_k(t) \le x\} = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{1}{2t}[h_t^{-1}(x)]^2} \frac{d}{dx} h_t^{-1}(x), \quad t \le T_{k-1}$$

Properties of the New Class (cont'd)

Transition density. Given two any instants $t < T \leq T_{k-1}$, the forward rate $F_k(T)$ conditional on $F_k(t)$ can be written as

$$F_k(T) = h\left(T, h_t^{-1}(F_k(t)) + W_T - W_t\right),$$

so that denoting by $p(t, y; T, x)$ the density of $F_k(T)$ conditional on $F_k(t) = y$ we have

$$\begin{aligned} Q^k\{F_k(T) \leq x | F_k(t) = y\} &= Q^k\left\{W_T - W_t \leq h_T^{-1}(x) - h_t^{-1}(y) | F_k(t) = y\right\} \\ &= \Phi\left(\frac{h_T^{-1}(x) - h_t^{-1}(y)}{\sqrt{T-t}}\right) \end{aligned}$$

$$p(t, y; T, x) = \frac{e^{-\frac{1}{2(T-t)}\left[h_T^{-1}(x) - h_t^{-1}(y)\right]^2}}{\sqrt{2\pi(T-t)}} \frac{d}{dx} h_T^{-1}(x)$$

Properties of the New Class (cont'd)

Caplet pricing. The price at time t of the caplet resetting in T_{k-1}, paying in T_k and with strike K is given by

$$\textbf{Cpl}(t, T_{k-1}, T_k, \tau_k, K) = \tau_k P(t, T_k) E^k\{[F_k(T_{k-1}) - K]^+ | \mathcal{F}_t\}$$

$$= \tau_k P(t, T_k) \int_{h^{-1}_{T_{k-1}}(K) - h_t^{-1}(F_k(t))}^{+\infty} \frac{h_{T_{k-1}}\left(h_t^{-1}(F_k(t)) + w\right)}{\sqrt{2\pi(T_{k-1} - t)}} e^{-\frac{w^2}{2(T_{k-1}-t)}} \, dw$$

$$- K\tau_k P(t, T_k) \Phi\left(\frac{h_t^{-1}(F_k(t)) - h^{-1}_{T_{k-1}}(K)}{\sqrt{T_{k-1} - t}}\right),$$

$$= \tau_k P(t, T_k) \left[F_k(t) - K + \int_0^K \Phi\left(\frac{h^{-1}_{T_{k-1}}(z) - h_t^{-1}(F_k(t))}{\sqrt{T_{k-1} - t}}\right) dz\right]$$

The New General Class: Importance and Usefulness

The models in this second class:

- Can be defined by taking into account the specific nature of forward rates (expiry time).
- Possess all the analytical tractability we commonly require: explicit SDE, known marginal and transition densities, explicit option prices at any time.
- Allow for quick checks of the evolution of the implied caplet volatility curve in the future.
- Allow for a consistent pricing of path-dependent derivatives.

A Particular Case: a Mixture of GBMs

We consider a linear combination of N driftless and perfectly correlated geometric Brownian motions:

$$F_k(t) = h(t, W_t) \quad \text{for each } 0 \leq t \leq T_{k-1},$$

$$h(t, w) = h_t(w) = \sum_{i=1}^{N} \psi_i \, e^{-\frac{1}{2}\beta_i^2 t + \beta_i w},$$

where $F_k(0)$, β_i's and ψ_i's are positive constants.

This function h fulfills our initial assumptions. Moreover, the initial condition imposes that

$$\sum_{i=1}^{N} \psi_i = F_k(0)$$

A Mixture of GBMs: the Associated Dynamics

Setting, for each i, $\lambda_i := \psi_i / F_k(0)$, we can write

$$F_k(t) = \sum_{i=1}^{N} \lambda_i Y_i(t), \quad dY_i(t) = Y_i(t)\beta_i \, dW_t, \quad Y_i(0) = F_k(0)$$

$$dF_k(t) = \sum_{i=1}^{N} \lambda_i Y_i(t)\beta_i \, dW_t = \sum_{i=1}^{N} \psi_i \beta_i e^{-\frac{1}{2}\beta_i^2 t + \beta_i h_t^{-1}(F_k(t))} \, dW_t$$

Hence,

$$dF_k(t) = F_k(t) \left[\sum_{i=1}^{N} \Lambda_i(t, F_k(t))\beta_i \right] dW_t$$

$$\Lambda_i(t, z) := \frac{\psi_i \, e^{-\frac{1}{2}\beta_i^2 t + \beta_i h_t^{-1}(z)}}{\sum_{j=1}^{N} \psi_j \, e^{-\frac{1}{2}\beta_j^2 t + \beta_j h_t^{-1}(z)}}$$

A Mixture of GBMs: Analytical Tractability

Marginal and transition densities. They are obtained from the general formulas by explicit calculation of $\frac{d}{dx}h_t^{-1}(x)$ and $\frac{d}{dx}h_T^{-1}(x)$.

Caplet pricing. Integration of the general formula leads to

$$\textbf{Cpl}(t, T_{k-1}, T_k, \tau_k, K) = \tau_k P(t, T_k) F_k(t)$$
$$\cdot \sum_{i=1}^{N} \Lambda_i(t, F_k(t)) \Phi\left(\frac{\beta_i(T_{k-1} - t) - h_{T_{k-1}}^{-1}(K) + h_t^{-1}(F_k(t))}{\sqrt{T_{k-1} - t}}\right)$$
$$- K \tau_k P(t, T_k) \Phi\left(\frac{h_t^{-1}(F_k(t)) - h_{T_{k-1}}^{-1}(K)}{\sqrt{T_{k-1} - t}}\right)$$

A Mixture of GBMs: Analytical Tractability (cont'd)

In particular, the caplet price at time $t = 0$ reduces to

$$\tau_k P(0,T_k)\left[\sum_{i=1}^{N}\psi_k\,\Phi\left(\frac{\beta_i T_{k-1} - h_{T_{k-1}}^{-1}(K)}{\sqrt{T_{k-1}}}\right) - K\Phi\left(-\frac{h_{T_{k-1}}^{-1}(K)}{\sqrt{T_{k-1}}}\right)\right]$$

Implied volatility curves. They typically show weird patterns (increasing and concave in the strike) which renders our GBM mixture model hardly suitable for calibration to market data.

However, we can resort to a slightly more general model, where we only require that $\psi_i\beta_i > 0$ for each i.

A General Dynamics à la Dupire

Assume that caplet prices are available for a continuum of strikes.

Denote by $C_k(K)$ the price of the caplet associated to F_k with strike K.

The following no-arbitrage conditions are assumed to hold:

- $C_k \in C^2((0,+\infty))$.
- $\lim_{x\to 0^+} C_k(x) = \tau_k P(0,T_k)F_k(0)$ and $\lim_{x\to +\infty} C_k(x) = 0$.
- $\lim_{x\to 0^+} \frac{dC_k}{dx}(x) = -\tau_k P(0,T_k)$ and $\lim_{x\to +\infty} x\frac{dC_k}{dx}(x) = 0$.
- $\frac{d^2C_k}{dx^2}(x) > 0$ for each $x > 0$, implying $-\tau_k P(0,T_k) < \frac{dC_k}{dx}(x) < 0$ for each $x > 0$.

Exact Calibration to Caplet Data

Proposition. *The function $h_{T_{k-1}}$ that is consistent with the given caplet prices is implicitly defined by*

$$h_{T_{k-1}}^{-1}(x) = -\sqrt{T_{k-1}}\,\Phi^{-1}\left(-\frac{\frac{dC_k}{dx}(x)}{\tau_k P(0,T_k)}\right), \quad x > 0$$

which is well defined due to the assumptions on C_k.

Proof. *Following Breeden and Litzenberger (1978), we get*

$$\begin{aligned}\frac{\partial}{\partial K}\textbf{Cpl}(0,T_{k-1},T_k,\tau_k,K) &= \tau_k P(0,T_k)\left[Q^k\{F_k(T_{k-1}) \le K\} - 1\right] \\ &= -\tau_k P(0,T_k)\Phi\left(-\frac{h_{T_{k-1}}^{-1}(K)}{\sqrt{T_{k-1}}}\right)\end{aligned}$$

Exact Calibration to Caplet Data (cont'd)

Corollary. *The function $h_{T_{k-1}}$ can be explicitly written as*

$$h_{T_{k-1}}(w) = \left(\frac{dC_k}{dx}\right)^{-1}\left(-\tau_k P(0,T_k)\Phi\left(-\frac{w}{\sqrt{T_{k-1}}}\right)\right), \quad w \in \mathbb{R}$$

The function $h_{T_{k-1}}$ is strictly positive, differentiable, increasing and with zero limit at minus infinity, $\lim_{w\to-\infty} h_{T_{k-1}}(w) = 0$. Moreover, the Q^k-expectation of $h_{T_{k-1}}(W_{T_{k-1}})$ is finite and equal to $F_k(0)$.

N.B. The value of $h_{T_{k-1}}$ in any point w can also be obtained by numerically solving, in the variable x,

$$w - h^{-1}_{T_{k-1}}(x) = 0$$

The Implied Forward Rate Value and Dynamics

Proposition. *The value of the forward rate F_k at any time $t < T_{k-1}$ that is consistent with $h_{T_{k-1}}$ is $F_k(t) = h_t(W_t)$ where*

$$h_t(w) = E^k[h_{T_{k-1}}(W_{T_{k-1}})|W_t = w]$$
$$= \int_0^{+\infty} \Phi\left(\frac{w + \sqrt{T_{k-1}}\,\Phi^{-1}\big(-\frac{1}{\tau_k P(0,T_k)}\frac{dC_k}{dz}(z)\big)}{\sqrt{T_{k-1}-t}}\right)dz$$

Corollary. *The dynamics of F_k that is consistent with the above forward rate value is*

$$dF_k(t) = \int_0^{+\infty} \frac{\exp\left\{-\frac{\left[h_t^{-1}(F_k(t))+\sqrt{T_{k-1}}\,\Phi^{-1}\left(-\frac{1}{\tau_k P(0,T_k)}\frac{dC_k}{dz}(z)\right)\right]^2}{2(T_{k-1}-t)}\right\}}{\sqrt{2\pi(T_{k-1}-t)}}\,dz\; dW_t$$

Future Caplet Prices and Implied Volatility Curves

To check the evolution of the implied volatility curves produced by the forward rate process in the future, we need to analytically price caplets at any future time $0 < t < T_{k-1}$.

From the general pricing formula, we obtain the following.

Proposition. *The price at time $t < T_{k-1}$ of the caplet with strike K is given by*

$$\textbf{Cpl}(t, T_{k-1}, T_k, \tau_k, K) = \tau_k P(t, T_k)$$
$$\cdot \left[F_k(t) - \int_0^K \Phi\left(\frac{\sqrt{T_{k-1}}\,\Phi^{-1}\big(-\frac{1}{\tau_k P(0,T_k)}\frac{dC_k}{dz}(z)\big) + h_t^{-1}(F_k(t))}{\sqrt{T_{k-1}-t}} \right) dz \right]$$

and can easily be calculated with numerical methods.

Parameterizing the Market Caplet Prices

The previous propositions can be restated for a general pricing function C_k coming from a flexible parametric distribution for $F_k(T_{k-1})$. For instance,

$$p_{T_{k-1}}(x) = \sum_{i=1}^{N} \lambda_i \frac{1}{x\sigma_i\sqrt{2\pi T_{k-1}}} \exp\left\{ -\frac{\left[\ln\frac{x}{F_k(0)} - \mu_i T_{k-1} + \frac{1}{2}\sigma_i^2 T_{k-1}\right]^2}{2\sigma_i^2 T_{k-1}} \right\}$$

with $\sum_{i=1}^{N} \lambda_i \, e^{\mu_i T_{k-1}} = 1$. In this case, we have

$$C_k(K) = \tau_k P(0, T_k) \sum_{i=1}^{N} \lambda_i e^{\mu_i T_{k-1}} \mathrm{Bl}\left(K\, e^{-\mu_i T_{k-1}}, F_k(0), \sigma_i\sqrt{T_{k-1}}\right)$$

Several advantages: i) less severe restrictions on the model parameters, ii) constant coefficients iii) rapid calculation of future caplet prices.