LOGNORMAL MIXTURE SMILE CONSISTENT OPTION PRICING

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Stylized facts

- Traders use the Black-Scholes formula to price plain-vanilla options.

- An option is priced through its *implied volatility*, the σ parameter to plug into the Black-Scholes formula to match the corresponding market price:

$$S_0 e^{-qT} \Phi\left(\frac{\ln(S_0/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$
$$- K e^{-rT} \Phi\left(\frac{\ln(S_0/K) + (r - q - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) = C(K,T)$$

- Implied volatilities vary with strike and maturity, namely $\sigma = \sigma(K,T)$. Implied volatility curves (surfaces) are usually skew-shaped or smile-shaped.

Stylized facts (cont'd)

- Consequence: the Black-Scholes model cannot consistently price all options in a market (the risk-neutral distribution is not lognormal).

- If volatilities were flat for each fixed maturity, we could assume

  $$dS_t = \mu S_t dt + \sigma_t S_t dW_t,$$

  where σ_t is deterministic, and recursively solve

  $$\int_0^{T_i} \sigma_t^2 dt = v_i^2 T_i \quad (v_i \text{ implied volatility for } T_i)$$

- However, more complex structures are present in real markets.

- We must resort to an alternative model:

  $$dS_t = \mu S_t dt + \varsigma_t S_t dW_t,$$

  where either $\varsigma_t = \varsigma_t(t, S_t)$ or ς_t is stochastic (driven by another BM).

Motivation of the paper

- To consistently price all (plain-vanilla) options in a given market, we must resort to an alternative model for the asset price S.

- A *good* alternative model
  - Has explicit dynamics, possibly with known marginal density.
  - Implies analytical formulas for European options.
  - Implies a good fitting of market data.
  - Implies a nice evolution of the volatility structures in the future.

- It would be *great* if the alternative model also
  - Had explicit transition densities.
  - Implied closed form formulas for a number of path-dependent derivatives (barriers, lookbacks,...).

The lognormal-mixture (LM) local-volatility (LV) model

Assume that the asset price risk-neutral drift rate is a deterministic function $\mu(t)$, and set $M(t) := \int_0^t \mu(s)\, ds$.

Consider N deterministic functions (volatilities) $\sigma_1, \dots, \sigma_N$, such that

- $\sigma_1, \dots, \sigma_N$ are continuous and bounded from below by positive constants.
- $\sigma_i(t) = \sigma_0 > 0$, for each $t \in [0, \varepsilon]$, $\varepsilon > 0$, and $i = 1, \dots, N$.

Let $S(0) = S_0 > 0$. Consider N lognormal densities at time t, $i = 1, \dots, N$,

$$p_t^i(y) = \frac{1}{y V_i(t)\sqrt{2\pi}} \exp\left\{ -\frac{\left[\ln \frac{y}{S_0} - M(t) + \frac{1}{2} V_i^2(t)\right]^2}{2 V_i^2(t)} \right\}, \quad V_i(t) := \sqrt{\int_0^t \sigma_i^2(u) du}$$

The LMLV model (cont'd)

Proposition. [Brigo and Mercurio, 2000] *If we set, for* $(t, y) > (0, 0)$,

$$\nu(t,y) = \sqrt{\frac{\sum_{i=1}^{N} \lambda_i \sigma_i^2(t) \frac{1}{V_i(t)} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{S_0} - M(t) + \frac{1}{2}V_i^2(t)\right]^2\right\}}{\sum_{i=1}^{N} \lambda_i \frac{1}{V_i(t)} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{S_0} - M(t) + \frac{1}{2}V_i^2(t)\right]^2\right\}}}$$

and $\nu(0, S_0) = \sigma_0$, *then the SDE*

$$dS(t) = \mu(t)S(t)\,dt + \nu(t, S(t))S(t)\,dW(t)$$

has a unique strong solution whose marginal density is

$$p_t(y) = \sum_{i=1}^{N} \lambda_i \frac{1}{yV_i(t)\sqrt{2\pi}} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{S_0} - M(t) + \frac{1}{2}V_i^2(t)\right]^2\right\}$$

The LMLV model: advantages and drawbacks

Advantages:

- Explicit marginal density.
- Explicit option prices (mixtures of Black-Scholes prices).
- Nice fitting to smile-shaped implied volatility curves and surfaces.
- Market completeness.

Drawbacks:

- Not so good fitting to skew-shaped implied volatility curves and surfaces.
- Unknown transition density.
- Future implied volatilities must be calculated numerically (Monte Carlo).

A lognormal-mixture uncertain-volatility (UV) model

We assume that the asset price dynamics under the risk neutral measure is

$$dS(t) = \begin{cases} S(t)[\mu(t)\,dt + \sigma_0\,dW(t)] & t \in [0, \varepsilon] \\ S(t)[\mu(t)\,dt + \eta(t)\,dW(t)] & t > \varepsilon \end{cases}$$

where η is a random variable that is independent of W and takes values in a set of N (given) deterministic functions:

$$\eta(t) = \begin{cases} \sigma_1(t) & \text{with probability } \lambda_1 \\ \sigma_2(t) & \text{with probability } \lambda_2 \\ \vdots & \vdots \\ \sigma_N(t) & \text{with probability } \lambda_N \end{cases}$$

The random value of η is drawn at time $t = \varepsilon$.

N.B. This model is similar in spirit to (but derived independently from) those of Alexander, Brintalos, and Nogueira (2003) and Gatarek (2003).

The LMUV model: advantages

A clear interpretation: the LMUV model is a Black-Scholes model where the asset volatility is unknown and one assumes different scenarios for it.

The LMUV model has the same advantages as the LMLV model:

- Explicit marginal density (mixture of lognormal densities).
- Explicit option prices (mixtures of Black-Scholes prices).
- Nice fitting to smile-shaped implied volatility curves and surfaces.
- It allows for a natural extension to the lognomal LIBOR market model.

In addition, the LMUV model is analytically tractable also after time 0, since, for $t > \varepsilon$, S follows a geometric Brownian motion. We thus have:

- Explicit transitions densities.
- Explicit prices for a number of path-dependent payoffs.

LMUV model vs LMLV model

Assume that the functions σ_i satisfy, for $t \geq \varepsilon$, the same assumptions as in the LMLV model. In particular, $\sigma_i(\varepsilon) = \sigma_0$, for $i = 1, \ldots, N$.

The marginal density of S at time t then coincides with that of the LMLV model, i.e.

$$p_t(y) = \sum_{i=1}^{N} \lambda_i \frac{1}{y V_i(t)\sqrt{2\pi}} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln \frac{y}{S_0} - M(t) + \tfrac{1}{2} V_i^2(t) \right]^2 \right\}$$

Under the LMUV model, the market is incomplete since the asset volatility is "stochastic".

However, the dynamics of S is directly given under the pricing measure: LMUV and LMLV European option prices coincide.

LMUV model vs LMLV model (cont'd)

Proposition. *Under the previous assumptions on the functions* σ_i, *the LMLV model is the projection of the LMUV model onto the class of local volatility models, in that (Derman and Kani, 1998)*

$$\nu^2(T, K) = E[\eta^2(T) | S(T) = K]$$

Proof. The equality follows from the definitions of $\eta(t)$ and $\nu(t, y)$ and a simple application of the Bayes rule.

A further analogy between the LMUV and LMLV models is that:

$$\mathsf{Corr}\big(\nu^2(t, S(t)), S(t)\big) = \mathsf{Corr}\big(\eta^2(t), S(t)\big) = 0$$

Calibration of the LMLV and LMUV models to FX volatility data: the market volatility matrix

	25Δ	50Δ	75Δ
1W	9.83%	9.45%	9.63%
2W	9.76%	9.40%	9.61%
1M	9.66%	9.25%	9.41%
2M	9.76%	9.40%	9.61%
3M	10.16%	9.85%	10.11%
6M	10.66%	10.40%	10.71%
9M	10.90%	10.65%	11.98%
1Y	10.99%	10.75%	11.09%
2Y	11.12%	10.85%	11.17%

Table 1: EUR/USD implied volatilities on 12 April 2002.

Calibration of the LMLV and LMUV models to FX volatility data: the calibrated volatility matrix

	25Δ	50Δ	75Δ
1W	9.55%	9.09%	9.55%
2W	9.66%	9.20%	9.67%
1M	9.76%	9.30%	9.76%
2M	10.06%	9.59%	10.07%
3M	10.32%	9.82%	10.35%
6M	10.84%	10.31%	10.89%
9M	11.12%	10.58%	11.19%
1Y	11.27%	10.73%	11.35%
2Y	11.39%	10.90%	11.53%

Table 2: Calibrated volatilities obtained through a suitable parametrization of the functions σ_i.

Calibration of the LMLV and LMUV models to FX volatility data: absolute errors

	25Δ	50Δ	75Δ
1W	-0.28%	-0.36%	-0.08%
2W	-0.10%	-0.20%	0.06%
1M	0.10%	0.05%	0.35%
2M	0.30%	0.19%	0.46%
3M	0.16%	-0.03%	0.24%
6M	0.18%	-0.09%	0.18%
9M	0.22%	-0.07%	0.21%
1Y	0.28%	-0.02%	0.26%
2Y	0.27%	0.05%	0.36%

Table 3: Differences between calibrated volatilities and market volatilities.

The pricing of a barrier option under the LMUV model

Assume $\mu(t) = r(t) - q(t)$. The price at time 0 of an up-and-out call with barrier $H > S_0$, strike K and maturity T is approximately (Lo-Lee, 2001)

$$1_{\{K<H\}} \sum_{i=1}^{N} \lambda_i \Bigg\{ S_0 e^{c_1+c_2+c_3} \left[\Phi\left(\frac{\ln \frac{S_0}{K} + c_1 + 2c_2}{\sqrt{2c_2}} \right) - \Phi\left(\frac{\ln \frac{S_0}{H} + c_1 + 2c_2}{\sqrt{2c_2}} \right) \right]$$

$$- K e^{c_3} \left[\Phi\left(\frac{\ln \frac{S_0}{K} + c_1}{\sqrt{2c_2}} \right) - \Phi\left(\frac{\ln \frac{S_0}{H} + c_1}{\sqrt{2c_2}} \right) \right] - H e^{c_3 + (\beta-1)(\ln \frac{S_0}{H} + c_1) + (\beta-1)^2 c_2}$$

$$\cdot \left[\Phi\left(\frac{\ln \frac{S_0}{H} + c_1 + 2(\beta-1)c_2}{\sqrt{2c_2}} \right) - \Phi\left(\frac{\ln \frac{S_0 K}{H^2} + c_1 + 2(\beta-1)c_2}{\sqrt{2c_2}} \right) \right]$$

$$+ K e^{c_3 + \beta(\ln \frac{S_0}{H} + c_1) + \beta^2 c_2} \left[\Phi\left(\frac{\ln \frac{S_0}{H} + c_1 + 2\beta c_2}{\sqrt{2c_2}} \right) - \Phi\left(\frac{\ln \frac{S_0 K}{H^2} + c_1 + 2\beta c_2}{\sqrt{2c_2}} \right) \right] \Bigg\}$$

where c_1, c_2, c_3 and β are integrals depending on $r(t)$, $q(t)$ and $\sigma_i(t)$.

Examples of FX barrier option prices under the LMLV and LMUV models

Type	LMLV	LMUV	BS
UOC(T=3M,K=1,H=1.05)	29	28	36
UOC(T=6M,K=1,H=1.08)	48	47	57
UOC(T=9M,K=1,H=1.10)	57	55	68
DOC(T=3M,K=1.02,H=0.95)	96	98	99
DOC(T=6M,K=1.07,H=0.98)	65	67	59
DOC(T=9M,K=1.10,H=0.90)	68	70	59

Table 4: Barrier option prices in basis points ($S_0 = 1$).

A simple extension of the LMUV model

Consider a new asset price dynamics under the risk neutral measure:

$$dS(t) = \begin{cases} S(t)[\mu(t)\,dt + \sigma_0\,dW(t)] & t \in [0,\varepsilon] \\ \mu(t)S(t)\,dt + \psi(t)[S(t) - \alpha e^{M(t)}]\,dW(t) & t > \varepsilon \end{cases}$$

where (ψ, α) is a random pair that is independent of W and takes values in the set of N (given) pairs of deterministic functions and real constants:

$$(\psi(t), \alpha) = \begin{cases} (\sigma_1(t), \alpha_1) & \text{with probability } \lambda_1 \\ (\sigma_2(t), \alpha_2) & \text{with probability } \lambda_2 \\ \vdots & \vdots \\ (\sigma_N(t), \alpha_N) & \text{with probability } \lambda_N \end{cases}$$

The random value of (ψ, α) is again drawn at time $t = \varepsilon$.

A simple extension of the LMUV model: features

A clear interpretation: the extended model is a displaced Black-Scholes model where both the asset volatility and the displacement are unknown. One then assumes different (joint) scenarios for them.

The extended model has the same advantages as the LMUV model:

- Explicit marginal density (mixture of displaced lognormal densities).
- Explicit option prices (mixtures of displaced Black-Scholes prices).
- Explicit transitions densities.
- Explicit (approximated) prices for barrier options.
- It allows for a natural extension to the lognomal LIBOR market model.

In addition, the extended model can lead to a

- Nice fitting to skew-shaped implied volatility curves and surfaces.

Application to the LIBOR market model

- Let $\mathcal{T} = \{T_0, \ldots, T_M\}$ be a set of times and $\{\tau_0, \ldots, \tau_M\}$ the corresponding year fractions: τ_k is the year fraction for (T_{k-1}, T_k). We set $T_{-1} := 0$.

- We consider a family of spanning forward rates with expiry T_{k-1} and maturity T_k, $k = 1, \ldots, M$:

$$F_k(t) := F(t; T_{k-1}, T_k) = \frac{P(t, T_{k-1}) - P(t, T_k)}{\tau_k P(t, T_k)}$$

  with $P(t, T)$ the time-t price of the zero-coupon bond with maturity T.

- We denote by Q^k the T_k-*forward measure*, i.e. the probability measure associated with the numeraire $P(\cdot, T_k)$.

- The forward rate F_k is a martingale under Q^k, $k = 1, \ldots, M$.

Application to the LIBOR market model

- In the BGM model, F_k evolves under Q^k according to:

$$dF_k(t) = \sigma_k(t) F_k(t)\, dZ_k(t), \quad t \leq T_{k-1}$$

where the "local" volatilities σ_k's are deterministic and Z_k is the k-th component of an M-dimensional Brownian motion.

- The BGM price of the caplet with payoff $\tau_k [F_k(T_{k-1}) - K]^+$ at time T_k coincides with that given by the corresponding Black formula.

- The BGM model leads to flat caplets smiles. We thus have to resort to an alternative forward LIBOR model, whose dynamics are given under the spot LIBOR measure Q^d, *i.e.* the measure associated with the discretely rebalanced bank account numeraire:

$$B_d(t) = \frac{P(t, T_{m-1})}{\prod_{j=0}^{m-1} P(T_{j-1}, T_j)}, \quad T_{m-2} < t \leq T_{m-1}$$

Application to the LIBOR market model

As in Gatarek (2003), we now assume that the forward rates F_k evolve, under the spot LIBOR measure Q^d, as

$$dF_k(t) = \sigma_k^I(t)(F_k(t)+\alpha_k^I)\sum_{j=\beta(t)}^{k}\frac{\tau_j\rho_{j,k}\sigma_j^I(t)(F_j(t)+\alpha_j^I)}{1+\tau_jF_j(t)}\,dt + \sigma_k^I(t)(F_k(t)+\alpha_k^I)\,dZ_k^d(t)$$

- $\beta(t) = m$ if $T_{m-2} < t \leq T_{m-1}$;
- $Z^d = (Z_1^d, \ldots, Z_M^d)$ is a Brownian motion with $dZ_j^d dZ_k^d = \rho_{j,k}\,dt$;
- I is a random variable that is independent of Z^d and takes values in the set $\{1, 2, \ldots, N\}$ with $Q^d(I = i) = \lambda_i$, $\lambda_i > 0$ and $\sum_{i=1}^N \lambda_i = 1$;
- σ_k^i are (given) deterministic functions, and α_k^i are (given) real constants.

Application to the LIBOR market model

Caplets pricing

The generic forward rate F_k evolves under its canonical measure Q^k according to

$$dF_k(t) = \sigma_k^I(t)(F_k(t) + \alpha_k^I)\, dZ_k(t)$$

which leads to

$$\textbf{Caplet}(0, T_{k-1}, T_{k-1}, \tau_k, K) = \tau_k P(0, T_k) \sum_{i=1}^{N} \lambda_i \text{Bl}(K + \alpha_k^i, F_k(0) + \alpha_k^i, V_k^i)$$

$$\text{Bl}(K, F, v) = F\Phi(d_1(K, F, v)) - K\Phi(d_2(K, F, v))$$

$$d_{1,2}(K, F, v) = \frac{\ln(F/K) \pm v^2/2}{v}$$

$$V_i^k = \sqrt{\int_0^{T_{k-1}} [\sigma_k^i(t)]^2\, dt}$$

Application to the LIBOR market model

Swaptions pricing

The forward swap rate $S_{\alpha,\beta}(t)$ at time t for the set of times $T_\alpha, \dots, T_\beta$ is defined by

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{k=\alpha+1}^{\beta} \tau_k P(t,T_k)}$$

which can also be written as

$$S_{\alpha,\beta}(t) = \frac{\sum_{k=\alpha+1}^{\beta} \tau_k P(t,T_k) F_k(t)}{\sum_{k=\alpha+1}^{\beta} \tau_k P(t,T_k)} =: \sum_{k=\alpha+1}^{\beta} \omega_k(t) F_k(t)$$

By definition, the forward swap rate $S_{\alpha,\beta}(t)$ is a martingale under the forward swap measure $Q^{\alpha,\beta}$. By Ito's lemma, we then have that, under $Q^{\alpha,\beta}$,

$$dS_{\alpha,\beta}(t) = \sum_{k=\alpha+1}^{\beta} \frac{\partial S_{\alpha,\beta}(t)}{\partial F_k(t)} \sigma_k^I(t)(F_k(t)+\alpha_k^I)\, dZ_k^{\alpha,\beta}(t) =: \sum_{k=\alpha+1}^{\beta} \gamma_k \, dZ_k^{\alpha,\beta}(t)$$

Application to the LIBOR market model

Neglecting the dependence of $\omega_k(t)$ on $F_j(t)$, we have

$$\begin{aligned}
\gamma_k(t) &\approx \frac{\tau_k P(t,T_k)\sigma_k^I(t)(F_k(t)+\alpha_k^I)}{\sum_{h=\alpha+1}^{\beta}\tau_h P(t,T_h)} \\
&= \frac{\tau_k P(t,T_k)\sigma_k^I(t)(F_k(t)+\alpha_k^I)}{\sum_{h=\alpha+1}^{\beta}\tau_h P(t,T_h)(F_h(t)+\alpha_h^I)} \, \frac{\sum_{h=\alpha+1}^{\beta}\tau_h P(t,T_h)(F_h(t)+\alpha_h^I)}{\sum_{h=\alpha+1}^{\beta}\tau_h P(t,T_h)} \\
&\approx \frac{\tau_k P(0,T_k)\sigma_k^I(t)(F_k(0)+\alpha_k^I)}{\sum_{h=\alpha+1}^{\beta}\tau_h P(0,T_h)(F_h(0)+\alpha_h^I)} \left[S_{\alpha,\beta}(t) + \frac{\sum_{h=\alpha+1}^{\beta}\tau_h P(0,T_h)\alpha_h^I}{\sum_{h=\alpha+1}^{\beta}\tau_h P(0,T_h)} \right] \\
&=: \gamma_k^I(t)\left[S_{\alpha,\beta}(t)+\eta_{\alpha,\beta}^I\right]
\end{aligned}$$

We denote by $\gamma_k^i(t)$ and $\eta_{\alpha,\beta}^i$, respectively, the values of $\gamma_k^I(t)$ and $\eta_{\alpha,\beta}^I$ when $I=i$.

Application to the LIBOR market model

Therefore, the dynamics of $S_{\alpha,\beta}(t)$ under $Q^{\alpha,\beta}$ (approximately) reads as

$$dS_{\alpha,\beta}(t) = \sum_{k=\alpha+1}^{\beta} \gamma_k^I(t) \left[S_{\alpha,\beta}(t) + \eta_{\alpha,\beta}^I\right] dZ_k^{\alpha,\beta}(t)$$

$$= \left[S_{\alpha,\beta}(t) + \eta_{\alpha,\beta}^I\right] \sum_{k=\alpha+1}^{\beta} \gamma_k^I(t)\, dZ_k^{\alpha,\beta}(t) = \gamma_{\alpha,\beta}^I(t) \left[S_{\alpha,\beta}(t) + \eta_{\alpha,\beta}^I\right] dW^{\alpha,\beta}(t)$$

where

$$\gamma_{\alpha,\beta}^I(t) = \sqrt{\sum_{k,h=\alpha+1}^{\beta} \gamma_k^I(t)\gamma_h^I(t)\rho_{k,h}}\,, \quad dW^{\alpha,\beta}(t) = \frac{\sum_{k=\alpha+1}^{\beta} \gamma_k^I(t)\, dZ_k^{\alpha,\beta}(t)}{\gamma_{\alpha,\beta}^I(t)}$$

($W^{\alpha,\beta}|I$ is a BM). Notice also that

$$X^I(t) := S_{\alpha,\beta}(t) + \eta_{\alpha,\beta}^I \quad \Rightarrow \quad dX^I(t) = \gamma_{\alpha,\beta}^I(t) X^I(t)\, dW^{\alpha,\beta}(t)$$

Application to the LIBOR market model

The price of a European swaption with maturity T_α and strike K (fixed rate payments on dates $T_{\alpha+1}, \dots, T_\beta$) is

$$
\begin{aligned}
&\textbf{Swptn}(0, \alpha, \beta, K, \omega) \\
&= \sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i)\, E^{\alpha,\beta}[(\omega S_{\alpha,\beta}(T_\alpha) - \omega K)^+] \\
&= \sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i)\, E^{\alpha,\beta}\{[\omega X^I(T_\alpha) - \omega(K + \eta^I_{\alpha,\beta})]^+\} \\
&= \sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i) \sum_{i=1}^{N} \lambda_i\, E^{\alpha,\beta}\{[\omega X^I(T_\alpha) - \omega(K + \eta^I_{\alpha,\beta})]^+ | I = i\}
\end{aligned}
$$

Application to the LIBOR market model

We thus obtain:

$$\textbf{Swptn}(0,\alpha,\beta,K,\omega)$$

$$= \sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i) \sum_{i=1}^{N} \lambda_i \mathrm{Bl}(K+\eta^i_{\alpha,\beta}, S_{\alpha,\beta}(0)+\eta^i_{\alpha,\beta}, \Gamma^i_{\alpha,\beta}, \omega)$$

where

$$\Gamma^i_{\alpha,\beta} = \sqrt{\int_0^{T_\alpha} [\gamma^i_{\alpha,\beta}(t)]^2 \, dt} = \sqrt{\sum_{k,h=\alpha+1}^{\beta} \rho_{k,h} \int_0^{T_\alpha} \gamma^i_k(t)\gamma^i_h(t)\, dt}$$

This swaption price is nothing but a mixture of adjusted Black's swaption prices.

A general extension of the LMUV model

Consider a new asset price dynamics under the risk neutral measure:

$$dS(t) = \begin{cases} S(t)[(r_0 - q_0)\, dt + \sigma_0\, dW(t)] & t \in [0, \varepsilon] \\ S(t)[(r(t) - q(t))\, dt + \chi(t)\, dW(t)] & t > \varepsilon \end{cases}$$

where (r, q, χ) is a random triplet that is independent of W and takes values in the set of N (given) triplets of deterministic functions:

$$(r(t), q(t), \chi(t)) = \begin{cases} (r_1(t), q_1(t), \sigma_1(t)) & \text{with probability } \lambda_1 \\ (r_2(t), q_2(t), \sigma_2(t)) & \text{with probability } \lambda_2 \\ \quad\vdots & \quad\vdots \\ (r_N(t), q_N(t), \sigma_N(t)) & \text{with probability } \lambda_N \end{cases}$$

The random value of (r, q, χ) is again drawn at time $t = \varepsilon$.

The general LMUV model: features

Again, a clear interpretation: the extended model is a Black-Scholes model where the asset volatility, the risk free rate and the dividend yield are unknown. One then assumes different (joint) scenarios for them.

The general LMUV model has the same advantages as the LMUV model:

- Explicit marginal density (mixture of lognormals with different means).
- Explicit option prices (mixtures of Black-Scholes prices).
- Explicit transitions densities.
- Explicit (approximated) prices for barrier options.

In addition, the extended model leads to an

- Almost perfect fitting to any (smile-shaped or skew-shaped) implied volatility curves and surfaces.

The general LMUV model: application to the FX options market

We must impose the following no-arbitrage conditions.

- Exact fitting to the domestic zero-coupon curve:

$$\sum_{i=1}^{N} \lambda_i e^{-\int_0^t r_i(u)\,du} = P(0,t)$$

- Exact fitting to the foreign zero-coupon curve:

$$\sum_{i=1}^{N} \lambda_i e^{-\int_0^t q_i(u)\,du} = P^f(0,t)$$

where $q_i = r_i^f$.

Calibration of the general LMUV model to FX volatility data: the market surface

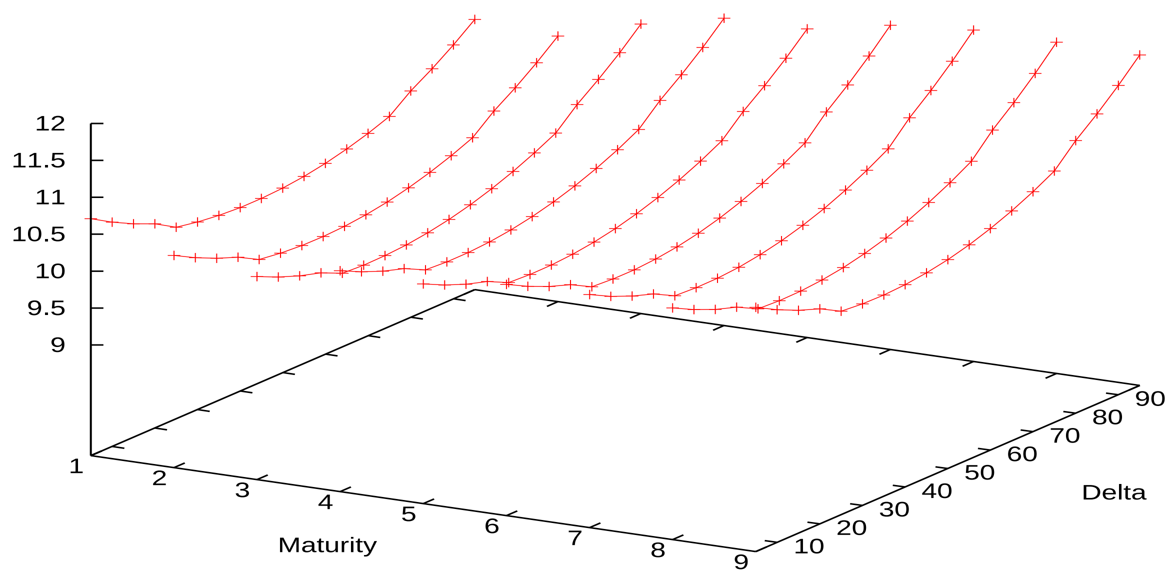

Figure 1: EUR/USD implied volatilities on 4 February 2003.

Calibration of the general LMUV model to FX volatility data: absolute errors

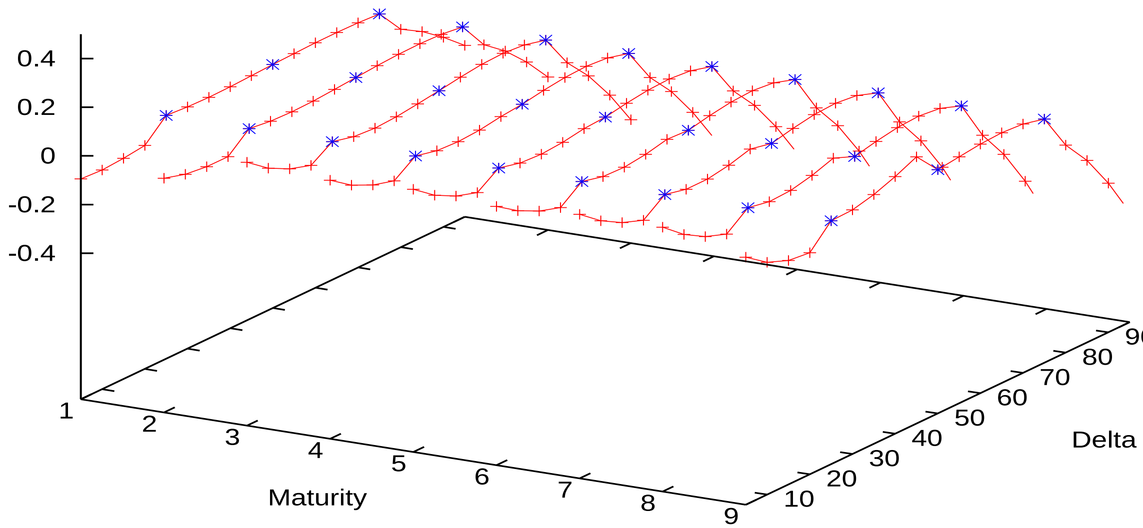

Figure 2: Differences between calibrated volatilities and market volatilities.

Evolution of the three-month implied volatility smile

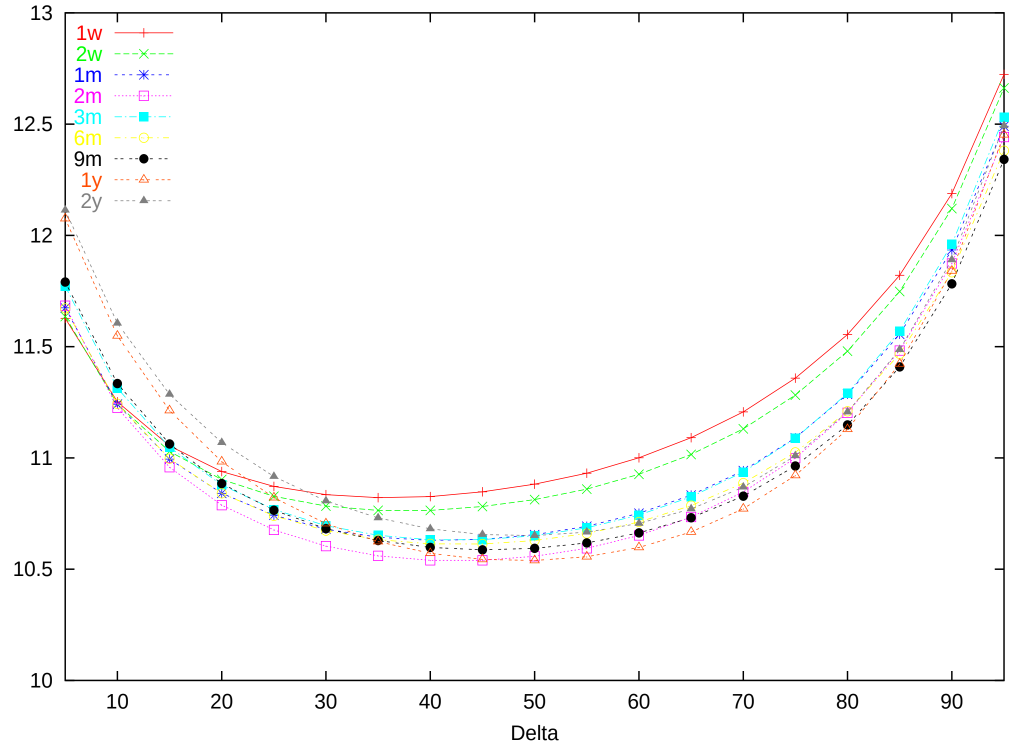

Figure 3: The three-month implied volatility smiles starting at different forward times $T \in \{1W, 2W, 1M, 2M, 3M, 6M, 9M, 1Y, 2Y\}$.

Conclusions

We have introduced a rather general uncertain volatility (and rate) model with special application to the FX options market. The model

- Has the tractability required (known marginal and transition densities, explicit European option prices);
- Prices analytically a number of exotic derivatives (barrier options, forward start options,...);
- Accommodates general implied volatility surfaces;
- Allows for Vega bucketing, i.e. for the calculation of the sensitivities with respect to each volatility quote.

A further extension is based on Markov chains (volatility and rates are drawn on several fixed dates).