Consistent Pricing of FX Options

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In the current markets, options with different strikes or maturities are usually priced with different implied volatilities. This stylized fact, which is commonly referred to as *smile* effect, can be accommodated by resorting to specific models, either for pricing exotic derivatives or for inferring implied volatilities for non quoted strikes or maturities. The former task is typically achieved by introducing alternative dynamics for the underlying asset price, whereas the latter is often tackled by means of statical adjustments or interpolations.

In this article, we deal with this latter issue and analyze a possible solution in a foreign exchange (FX) option market. In such a market, in fact, there are only three active quotes for each market maturity (the 0Δ straddle, the risk reversal and the vega-weighted butterfly), thus presenting us with the problem of a consistent determination of the other implied volatilities.

FX brokers and market makers typically address this issue by using an empirical procedure, also named Vanna-Volga (VV), to construct the whole smile for a given maturity. Volatility quotes are then provided in terms of the option's Δ, for ranges from the 5Δ put to the 5Δ call.

In the following, we will review this market procedure for a given currency. In particular, we will derive closed-form formulas so as to render its construction more explicit. We will then test the robustness (in a static sense) of the resulting smile, in that changing consistently the three initial pairs of strike and volatility produces eventually the same implied volatility curve. We will also show that the same procedure applied to European-style claims is consistent with static-replication results and consider, as an example, the practical case of a quanto European option. We will finally prove that the market procedure can also be justified in dynamical terms, by defining a hedging strategy that is locally replicating and self-financing.

2 A brief description of the FX option market

In the FX option market, the volatility matrix is built according to the sticky Delta rule. The underlying assumption is that options are priced depending on their Delta, so that when the underlying asset price moves and the Delta of an option changes accordingly, a different implied volatility has to be plugged into the pricing formula.

The FX option market is very liquid, up to relatively long dated expiries (2 years, at least for the EUR/USD exchange rate). The at-the-money (ATM) volatility is readily available, and the risk reversal (RR) for 25Δ call and put and the (vega-weighted) butterfly (VWB) with 25Δ wings are also commonly traded.[1] From these data one can easily infer three basic implied volatilities, from which one can then build the entire smile for the range running from a 5Δ put to a 5Δ call according to the method we shall outline below.

We denote by S_t the value of a given exchange rate at time t and assume constant domestic and foreign risk-free rates, which will be denoted respectively by r^d and r^f. We then consider a market maturity T and define the related quotes in the following.

The ATM volatility quoted in the FX market is that of a 0Δ straddle, whose strike, for each given expiry, is chosen so that a put and a call have the same Δ but with different signs (no Δ hedge is needed when trading this straddle).

Denoting by σ_{ATM} the ATM volatility for the expiry T, the ATM strike K_{ATM} must then satisfy

$$e^{-r^f T}\Phi\left(\frac{\ln\frac{S_0}{K_{ATM}} + (r^d - r^f + \frac{1}{2}\sigma_{ATM}^2)T}{\sigma_{ATM}\sqrt{T}}\right) = e^{-r^f T}\Phi\left(-\frac{\ln\frac{S_0}{K_{ATM}} + (r^d - r^f + \frac{1}{2}\sigma_{ATM}^2)T}{\sigma_{ATM}\sqrt{T}}\right)$$

where Φ denotes the cumulative standard normal distribution function. Straightforward algebra leads to:

$$K_{ATM} = S_0\, e^{(r^d - r^f + \frac{1}{2}\sigma_{ATM}^2)T} \tag{1}$$

The RR is a typical structure where one buys a call and sells a put with a symmetric Δ. The RR is quoted as the difference between the two implied volatilities, $\sigma_{25\Delta c}$ and $\sigma_{25\Delta p}$ to plug into the Black and Scholes formula for the call and the put respectively. Denoting such a price, in volatility terms, by σ_{RR}, we have:[2]

$$\sigma_{RR} = \sigma_{25\Delta c} - \sigma_{25\Delta p} \tag{2}$$

The VWB is built up by selling an ATM straddle and buying a 25Δ strangle. To be Vega-weighted, the quantity of the former has to be smaller than the quantity of the latter, since the Vega of the straddle is greater than the Vega of the strangle. The butterfly's price in volatility terms, σ_{VWB}, is then defined by:

$$\sigma_{VWB} = \frac{\sigma_{25\Delta c} + \sigma_{25\Delta p}}{2} - \sigma_{ATM} \tag{3}$$

For the given expiry T, the two implied volatilities $\sigma_{25\Delta c}$ and $\sigma_{25\Delta p}$ can be immediately identified by solving a linear system. We obtain:

$$\sigma_{25\Delta c} = \sigma_{ATM} + \sigma_{VWB} + \frac{1}{2}\sigma_{RR} \tag{4}$$

[1] We drop the "%" sign after the level of the Δ, in accordance to the market jargon. Therefore, a 25Δ call is a call whose Delta is 0.25. Analogously, a 25Δ put is one whose Delta is -0.25.

[2] A positive σ_{RR} means that the call is favored in that its implied volatility is higher than the implied volatility of the put; a negative number implies the opposite.

$$\sigma_{25\Delta p} = \sigma_{ATM} + \sigma_{VWB} - \tfrac{1}{2}\sigma_{RR} \tag{5}$$

The two strikes corresponding to the 25Δ put and 25Δ call can be derived, after straightforward algebra, by remembering their respective definitions. For instance, for a 25Δ put we must have that

$$-e^{-r^f T}\Phi\left(-\frac{\ln\frac{S_0}{K_{25\Delta p}} + (r^d - r^f + \frac{1}{2}\sigma^2_{25\Delta p})T}{\sigma_{25\Delta p}\sqrt{T}}\right) = -0.25$$

which immediately leads to

$$K_{25\Delta p} = S_0\, e^{-\alpha\sigma_{25\Delta p}\sqrt{T} + (r^d - r^f + \frac{1}{2}\sigma^2_{25\Delta p})T} \tag{6}$$

where $\alpha := -\Phi^{-1}(\frac{1}{4}e^{r^f T})$ and Φ^{-1} is the inverse normal distribution function. Similarly, one also gets:

$$K_{25\Delta c} = S_0\, e^{\alpha\sigma_{25\Delta c}\sqrt{T} + (r^d - r^f + \frac{1}{2}\sigma^2_{25\Delta c})T} \tag{7}$$

We stress that, for typical market parameters and for maturities up to two years, $\alpha > 0$ and[3]

$$K_{25\Delta p} < K_{ATM} < K_{25\Delta c}$$

In the following section, we will explain how to use the basic implied volatilities, and the related strikes, to consistently infer the entire smile for the given expiry T. To this end, we will work with the same type of options (*e.g.* calls), directly considering their market prices (instead of volatilities).

To lighten the notation and simplify future formulas, we will denote the quoted strikes (for the given maturity T) by K_i, $i = 1, 2, 3$, $K_1 < K_2 < K_3$,[4] and set $\mathcal{K} := \{K_1, K_2, K_3\}$. The related (market) option prices, respectively denoted by $C^{\text{MKT}}(K_1)$, $C^{\text{MKT}}(K_2)$ and $C^{\text{MKT}}(K_3)$, are assumed to satisfy the standard no-arbitrage conditions.

3 The VV empirical market procedure

Consider a European call option with maturity T and strike K, whose Black and Scholes price, at time t, is denoted by $C^{\text{BS}}(t; K)$,

$$C^{\text{BS}}(t;K) = S_t\, e^{-r^f\tau}\Phi\left(\frac{\ln\frac{S_t}{K} + (r^d - r^f + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - Ke^{-r^d\tau}\Phi\left(\frac{\ln\frac{S_t}{K} + (r^d - r^f - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \tag{8}$$

where $\tau := T - t$, and σ is a given volatility parameter.

It is well known that under the Black-Scholes (1973) (BS) model, the call payoff can be replicated by a dynamic Δ-hedging strategy, whose initial value (comprehensive of the bank account part) matches the option price (8). In real financial markets, however, volatility

[3]For long maturities, it is market practice to consider the forward exchange rate as the ATM strike.

[4]K_1, K_2 and K_3 replace respectively $K_{25\Delta p}$, K_{ATM} and $K_{25\Delta c}$.

is stochastic and traders hedge the associated risk by constructing Vega-neutral portfolios. Given the specific nature of the FX options market, portfolios can also be constructed so as to match partial derivatives up to the second order, so that, by Ito's lemma, we have a perfect hedge in an infinitesimal time interval, see also Section 9 below.

The empirical procedure is based on deriving such a hedging portfolio for the above call with maturity T and strike K. Precisely, we want to find time-t weights $x_1(t;K)$, $x_2(t;K)$ and $x_3(t;K)$ such that the resulting portfolio of European calls with maturity T and strikes K_1, K_2 and K_3, respectively, hedges the price variations of the call with maturity T and strike K, up to the second order in the underlying and the volatility. Assuming a Δ-hedged position and given that, in the BS world, portfolios of plain-vanilla options (with the same maturity) that are Vega neutral are also Gamma neutral, the weights $x_1(t;K)$, $x_2(t;K)$ and $x_3(t;K)$ can be found by imposing that the "replicating" portfolio has the same Vega, dVegadVol (volga) and dVegadSpot (vanna) as the call with strike K, namely

$$\begin{aligned} \frac{\partial C^{\text{BS}}}{\partial \sigma}(t;K) &= \sum_{i=1}^{3} x_i(t;K) \frac{\partial C^{\text{BS}}}{\partial \sigma}(t;K_i) \\ \frac{\partial^2 C^{\text{BS}}}{\partial^2 \sigma}(t;K) &= \sum_{i=1}^{3} x_i(t;K) \frac{\partial^2 C^{\text{BS}}}{\partial^2 \sigma}(t;K_i) \\ \frac{\partial^2 C^{\text{BS}}}{\partial \sigma \partial S_t}(t;K) &= \sum_{i=1}^{3} x_i(t;K) \frac{\partial^2 C^{\text{BS}}}{\partial \sigma \partial S_t}(t;K_i) \end{aligned} \tag{9}$$

Denoting by $\mathcal{V}(t;K)$ the time-t Vega of a European option with (maturity T and) strike K,

$$\begin{aligned} \mathcal{V}(t;K) &= \frac{\partial C^{\text{BS}}}{\partial \sigma}(t;K) = S_t\, e^{-r^f \tau} \sqrt{\tau} \varphi(d_1(t;K)) \\ d_1(t;K) &= \frac{\ln \frac{S_t}{K} + (r^d - r^f + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \\ \varphi(x) &= \Phi'(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} \end{aligned} \tag{10}$$

and calculating the second order derivatives

$$\begin{aligned} \frac{\partial^2 C^{\text{BS}}}{\partial^2 \sigma}(t;K) &= \frac{\mathcal{V}(t;K)}{\sigma}\, d_1(t;K) d_2(t;K) \\ \frac{\partial^2 C^{\text{BS}}}{\partial \sigma \partial S_t}(t;K) &= -\frac{\mathcal{V}(t;K)}{S_t \sigma \sqrt{\tau}}\, d_2(t;K) \\ d_2(t;K) &= d_1(t;K) - \sigma\sqrt{\tau} \end{aligned}$$

we can prove the following.

Proposition 3.1. *The system (9) admits always a unique solution, which is given by*

$$
\begin{aligned}
x_1(t;K) &= \frac{\mathcal{V}(t;K)}{\mathcal{V}(t;K_1)} \frac{\ln\frac{K_2}{K}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_1}}\\
x_2(t;K) &= \frac{\mathcal{V}(t;K)}{\mathcal{V}(t;K_2)} \frac{\ln\frac{K}{K_1}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_2}}\\
x_3(t;K) &= \frac{\mathcal{V}(t;K)}{\mathcal{V}(t;K_3)} \frac{\ln\frac{K}{K_1}\ln\frac{K}{K_2}}{\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}}
\end{aligned}
\tag{11}
$$

In particular, if $K = K_j$ then $x_i(t;K) = 1$ for $i = j$ and zero otherwise.

Proof. See the appendix. □

4 The resulting option price

We can now proceed to the definition of an option price that is consistent with the market prices of the basic options.

A "smile-consistent" price for the call with strike K is obtained by adding to the BS price the cost of implementing the above hedging strategy at prevailing market prices. In formulas, for $t = 0$,

$$
C(K) = C^{\mathrm{BS}}(K) + \sum_{i=1}^{3} x_i(K)[C^{\mathrm{MKT}}(K_i) - C^{\mathrm{BS}}(K_i)] \tag{12}
$$

where, to lighten the notation, the dependence on the valuation time t is hereafter omitted when zero.[5]

The new option price is thus defined by adding to the "flat smile" BS price the cost difference of the hedging portfolio induced by the market implied volatilities with respect to the constant volatility σ. Robustness and consistency results for the option price (12) are provided below.

When $K = K_j$ we clearly have that $C(K_j) = C^{\mathrm{MKT}}(K_j)$, since $x_i(K) = 1$ for $i = j$ and zero otherwise. Therefore, (12) defines nothing but a rule for either interpolating or extrapolating prices from the three option quotes $C^{\mathrm{MKT}}(K_1)$, $C^{\mathrm{MKT}}(K_2)$ and $C^{\mathrm{MKT}}(K_3)$.

A market implied volatility curve can then be constructed by inverting (12), for each considered K, through the BS formula. An example of such a curve is provided in Figure 1, where we plot implied volatilities both against strikes and against put Deltas. We use the following EUR/USD data as of 1 July 2005: $T = 3m(= 94/365y)$, $S_0 = 1.205$, $\sigma_{ATM} = 9.05\%$, $\sigma_{RR} = -0.50\%$, $\sigma_{VWB} = 0.13\%$, which lead to $\sigma_{25\Delta c} = 8.93\%$, $\sigma_{50\Delta c} = 9.05\%$, $\sigma_{25\Delta p} = 9.43\%$, $K_{\mathrm{ATM}} = 1.2114$, $K_{25\Delta p} = 1.1733$ and $K_{25\Delta c} = 1.2487$. See also Tables 1 and 2.

[5] This price depends on the volatility parameter σ. In practice, the typical choice is to set $\sigma = \sigma_{ATM}$.

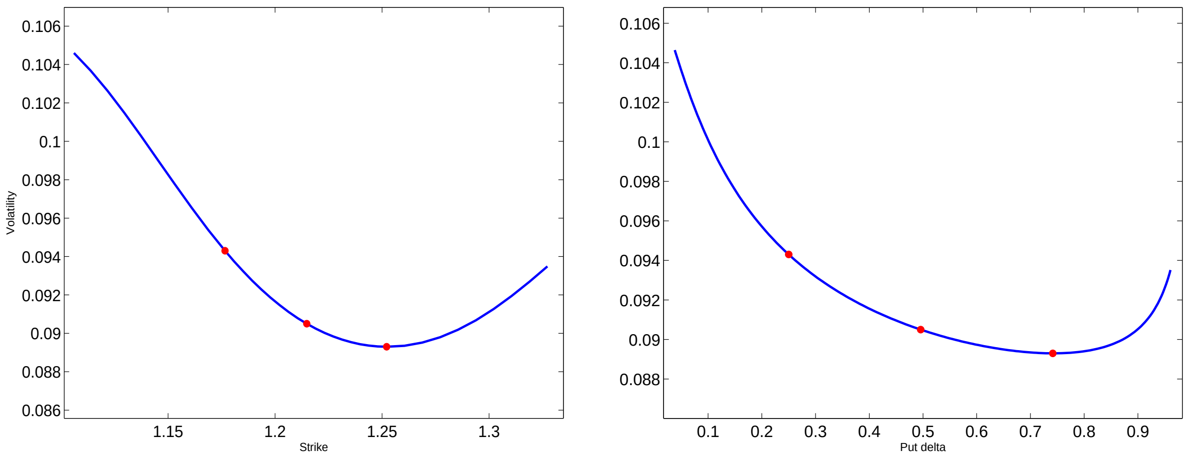

Figure 1: EUR/USD implied volatilities plotted both against strikes and against Deltas, where the three market basic quotes are highlighted.

The option price $C(K)$, as a function of the strike K, satisfies the following (no-arbitrage) conditions:

i) $C \in C^2((0, +\infty))$;

ii) $\lim_{K\to 0^+} C(K) = S_0\, e^{-r^f T}$ and $\lim_{K\to +\infty} C(K) = 0$;

iii) $\lim_{K\to 0^+} \frac{dC}{dK}(K) = -e^{-r^d T}$ and $\lim_{K\to +\infty} K \frac{dC}{dK}(K) = 0$.

The second and third properties, which are trivially satisfied by $C^{\mathrm{BS}}(K)$, follow from the fact that, for each i, both $x_i(K)$ and $dx_i(K)/dK$ go to zero for $K \to 0^+$ or $K \to +\infty$.

To avoid arbitrage opportunities, the option price $C(K)$ should also be a convex function of the strike K, *i.e.* $\frac{d^2C}{dK^2}(K) > 0$ for each $K > 0$. This property, which is not true in general,[6] holds however for typical market parameters, so that (12) leads indeed to prices that are arbitrage-free in practice.

5 An approximation for implied volatilities

The above definition of option price, combined with our analytical formula (11) for the weights, allows for the derivation of a straightforward approximation for the implied volatility associated to (11). This is described in the following.

Proposition 5.1. *The implied volatility $\sigma(K)$ for the above option with price $C(K)$ is approximately given by*

$$\sigma(K) \approx \sigma_1(K) := \frac{\ln \frac{K_2}{K} \ln \frac{K_3}{K}}{\ln \frac{K_2}{K_1} \ln \frac{K_3}{K_1}} \sigma_{25\Delta p} + \frac{\ln \frac{K}{K_1} \ln \frac{K_3}{K}}{\ln \frac{K_2}{K_1} \ln \frac{K_3}{K_2}} \sigma_{\mathrm{ATM}} + \frac{\ln \frac{K}{K_1} \ln \frac{K}{K_2}}{\ln \frac{K_3}{K_1} \ln \frac{K_3}{K_2}} \sigma_{25\Delta c} \qquad (13)$$

[6] One can actually find cases where the inequality is violated for some strike K.

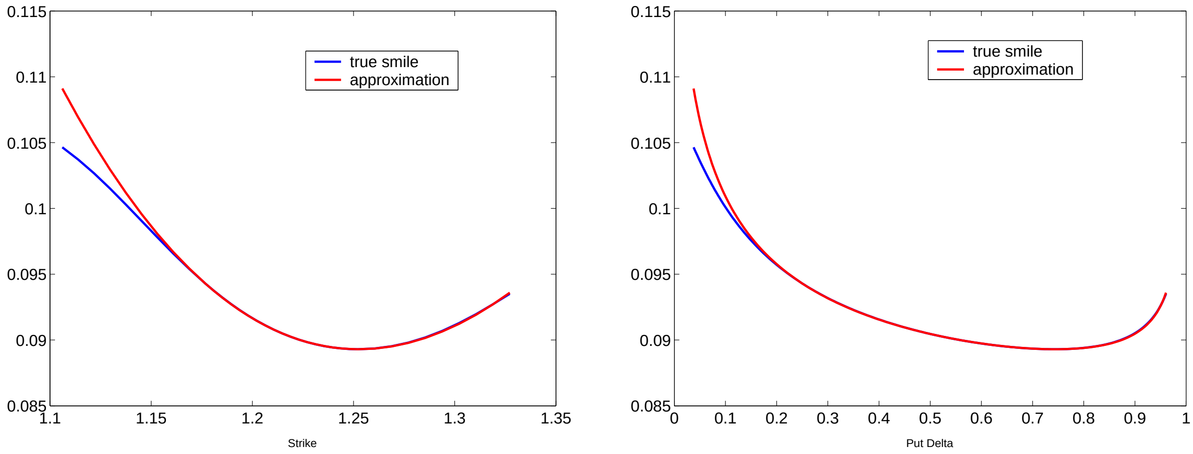

Figure 2: EUR/USD implied volatilities and their approximations, plotted both against strikes and against Deltas.

Proof. See the appendix. □

The implied volatility $\sigma(K)$ can thus be approximated by a linear combination of the basic volatilities, with combinators $y_i(K)$ that sum up to one (as tedious but straightforward algebra shows). It is also easily seen that the approximation is a quadratic function of $\ln K$, so that one can resort to a simple parabolic interpolation when log coordinates are used.

A graphical representation of the goodness of the approximation (13) is shown in Figure 2, where we use the same EUR/USD data as for Figure 1.

The approximation (13) is extremely accurate inside the interval $[K_1, K_3]$. The wings, however, tend to be overvalued. In fact, being the functional form quadratic in the log-strike, the no-arbitrage conditions derived by Lee (2004) for the asymptotic value of implied volatilities are here violated. This drawback is addressed by a second, more precise, approximation, which is asymptotically constant at extreme strikes.

Proposition 5.2. *The implied volatility $\sigma(K)$ can be better approximated as follows:*

$$\sigma(K) \approx \sigma_2(K) := \sigma + \frac{-\sigma + \sqrt{\sigma^2 + d_1(K)d_2(K)(2\sigma D_1(K) + D_2(K))}}{d_1(K)d_2(K)}, \tag{14}$$

where

$$D_1(K) := \frac{\ln\frac{K_2}{K}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_1}}\sigma_{25\Delta p} + \frac{\ln\frac{K}{K_1}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_2}}\sigma_{\mathrm{ATM}} + \frac{\ln\frac{K}{K_1}\ln\frac{K}{K_2}}{\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}}\sigma_{25\Delta c} - \sigma = \sigma_1(K) - \sigma$$

$$D_2(K) := \frac{\ln\frac{K_2}{K}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_1}}d_1(K_1)d_2(K_1)(\sigma_{25\Delta p} - \sigma)^2 + \frac{\ln\frac{K}{K_1}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_2}}d_1(K_2)d_2(K_2)(\sigma_{\mathrm{ATM}} - \sigma)^2 + \frac{\ln\frac{K}{K_1}\ln\frac{K}{K_2}}{\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}}d_1(K_3)d_2(K_3)(\sigma_{25\Delta c} - \sigma)^2$$

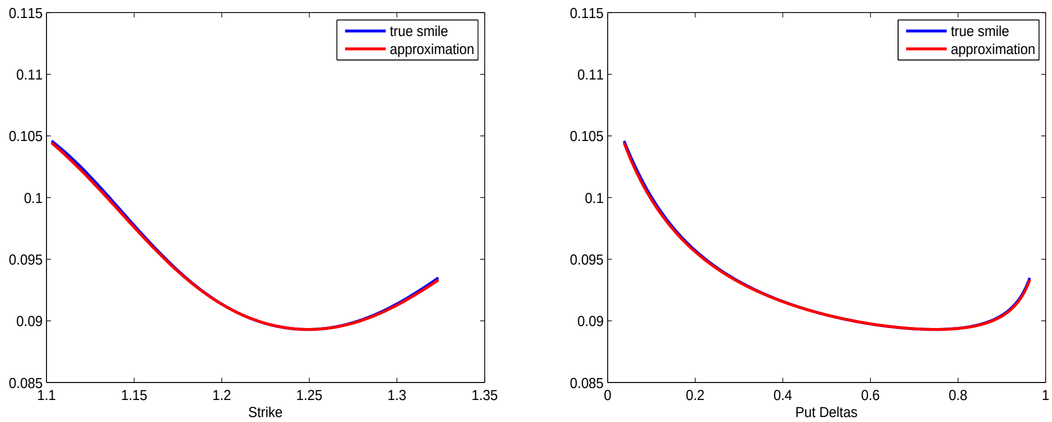

Figure 3: EUR/USD implied volatilities and their approximations, plotted both against strikes and against Deltas.

and

$$d_1(x) = \frac{\ln \frac{S_0}{x} + (r^d - r^f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2(x) = d_1(x) - \sigma\sqrt{T}, \quad x \in \{K, K_1, K_2, K_3\}$$

Proof. See the appendix. □

As we can see from Figure 3, the approximation (14) is extremely accurate also in the wings. Its only drawback is that it may not be defined due to the presence of a square-root term. The radicand, however, is positive in most practical applications.

6 A first consistency result for the price C(K)

We now state two important consistency results that hold for the option price (11) and that give further support to the above empirical procedure.

The first result is as follows. One may wonder what happens if we apply our curve construction method when starting from three other strikes whose associated prices coincide with those coming from formula (12). Clearly, for our procedure to be robust, we would want the two curves to exactly coincide.

In fact, consider a new set of strikes $\mathcal{H} := \{H_1, H_2, H_3\}$, and denote the previous weights $x_i(K)$ by $x_i(K;\mathcal{K})$ to stress the dependence on the set of initial strikes. Analogously, $x_i(K;\mathcal{H})$ will denote the weights for the strike K that are derived from the new set of strikes $\mathcal{H}$. The option price for each H_i is, by assumption, equal to that coming from (12), *i.e.*

$$C^{\mathcal{H}}(H_i) = C^{\mathcal{K}}(H_i) = C^{\text{BS}}(H_i) + \sum_{j=1}^{3} x_j(H_i;\mathcal{K})[C(K_j) - C^{\text{BS}}(K_j)] \qquad (15)$$

where the superscripts $\mathcal{H}$ and $\mathcal{K}$ highlight the set of strikes the pricing procedure is based on. For a generic strike K, the option price associated to $\mathcal{H}$ is defined, analogously to (12), by

$$C^{\mathcal{H}}(K) = C^{\text{BS}}(K) + \sum_{j=1}^{3} x_j(K; \mathcal{H})[C^{\mathcal{H}}(H_j) - C^{\text{BS}}(H_j)].$$

Proposition 6.1. *The call prices based on $\mathcal{H}$ coincide with those based on $\mathcal{K}$, namely, for each strike K,*

$$C^{\mathcal{H}}(K) = C^{\mathcal{K}}(K) \tag{16}$$

Proof. See the appendix. □

7 A second consistency result for the price C(K)

A second consistency result that can be proven for the option price (11) concerns the pricing of European-style derivatives and their static replication. To this end, assume that h is a real function that is defined on $[0, \infty)$, is well behaved at infinity and is twice differentiable in the sense of distributions. Given the *simple claim* with payoff $h(S_T)$ at time T, we denote by V its price at time 0, when taking into account the smile effect. By Carr and Madan (1998), we have:

$$V = e^{-r^d T} h(0) + S_0\, e^{-r^f T} h'(0) + \int_0^{+\infty} h''(x) C(x)\, dx$$

The same reasoning adopted before for the construction of the implied volatility curve can be applied to the general payoff $h(S_T)$. We can thus construct a portfolio of European calls with maturity T and strikes K_1, K_2 and K_3, such that the portfolio has the same Vega, dVegadVol and dVegadSpot as the given derivative. Denoting by V^{BS} the claim price under the Black and Scholes (1973) model, this is achieved by finding weights x_1^h, x_2^h and x_3^h such that

$$\begin{aligned}
\frac{\partial V^{\text{BS}}}{\partial \sigma} &= \sum_{i=1}^{3} x_i^h \frac{\partial C^{\text{BS}}}{\partial \sigma}(K_i) \\
\frac{\partial^2 V^{\text{BS}}}{\partial^2 \sigma} &= \sum_{i=1}^{3} x_i^h \frac{\partial^2 C^{\text{BS}}}{\partial^2 \sigma}(K_i) \\
\frac{\partial^2 V^{\text{BS}}}{\partial \sigma \partial S_0} &= \sum_{i=1}^{3} x_i^h \frac{\partial^2 C^{\text{BS}}}{\partial \sigma \partial S_0}(K_i)
\end{aligned}$$

which always exist unique, as already proved in Proposition 3.1. We can then define a new (smile consistent) price for our derivative as

$$\bar{V} = V^{\text{BS}} + \sum_{i=1}^{3} x_i^h [C(K_i) - C^{\text{BS}}(K_i)] \tag{17}$$

Proposition 7.1. *The claim price that is consistent with the option prices C is equal to the claim price that is obtained by adjusting its Black and Scholes price by the cost difference of the hedging portfolio when using market prices $C(K_i)$ instead of the constant volatility prices $C^{BS}(K_i)$. In formulas*

$$\bar{V} = V$$

Proof. See the appendix. □

This proposition states a clear consistency result for (European-style) simple claims. In fact, if we calculate the hedging portfolio for the claim under flat volatility and add to the claim price (calculated with the Black and Scholes model) the cost difference of the hedging portfolio (market price minus constant volatility price), we exactly retrieve the claim price as obtained through the risk-neutral density implied by the call option prices that are consistent with the market smile.

This useful result will be applied in the following section to the specific case of a quanto option.

8 An example: smile consistent pricing of a quanto option

A quanto option is a derivative paying out at maturity T the amount $[\omega(S_T - X)]^+$ in foreign currency, which is equivalent to $[\omega(S_T - X)]^+ S_T$ in domestic currency, where $\omega = 1$ for a call and $\omega = -1$ for a put. Standard arguments on static replication imply that quanto call and put prices can be written in terms of plain-vanilla call and put prices as follows

$$\begin{aligned} \mathrm{QCall}(T, X) &= 2\int_X^{+\infty} C(K)\,dK + XC(X) \\ \mathrm{QPut}(T, X) &= XP(X) - 2\int_0^X P(K)\,dK \end{aligned} \tag{18}$$

where $P(X)$ is the put price with strike X and maturity T, *i.e.* $P(X) = C(X) - S_0\, e^{-r^f T} + X\, e^{-r^d T}$.

We now verify, with real market data, that the quanto option prices (18) equal the prices (17) coming from hedging arguments. To this end we use the market data as of July 1^{st}, 2005, as reported in Tables 1 and 2.

Our calculations are reported in Table 3, where quanto option prices calculated with hedging arguments, *i.e.* with formula (17), are compared with the static replication prices (18) that are obtained by using 500 and 3000 steps and, respectively, a constant strike step of 0.15% and 0.25%. [7] The percentage differences between these prices are also shown.

[7] The integrals in (18) can of course be calculated with more efficient procedures. Here, however, we only want to show numerically the correctness of our pricing procedure.

Expiry		USD discount factor	EUR discount factor
3m:	03/10/2005	0.9902752	0.9945049
1y:	03/07/2006	0.9585801	0.9785056

Table 1: Market data as of July 1, 2005.

Delta	3M		1Y	
25Δ Put	1.1733	9.43%	1.1597	9.65%
ATM	1.2114	9.05%	1.2355	9.40%
25Δ Call	1.2487	8.93%	1.3148	9.43%

Table 2: Strikes and volatilities corresponding to the three main Delta's, as of July 1, 2005.

The purpose of this example is also to show that quanto option prices can be derived, consistently with the market smile, by using only three European options and not a continuum of strikes, as implied by (18).

9 Robustness of the Pricing Procedure

We conclude the article by motivating the empirical pricing procedure also in dynamical terms.

The apparently arbitrary approach of zeroing partial derivatives of BS prices up to the second order can be justified by the fact that the BS model is still a benchmark in the valuation of an option book. There are several reasons for this fact, apart from the obvious historical one: i) ease of implementation ii) clear and intuitive meaning of the model parameters; iii) readily available sensitivities; and iv) possibility of explicit formulas for most payoffs. No other model possesses all these features at the same time.[8] Actually, it is not such a weird practice to run an FX option book by revaluating and hedging it according to a flat-smile BS model, though the ATM volatility is continuously updated to the trading market level.[9]

We now prove that if European options are all valued with the same (stochastic) implied volatility (let's say the ATM volatility), the value changes of the hedging portfolio locally tracks those of the given call. To this end, we consider a generic time t and assume Ito-like dynamics for the volatility $\sigma = \sigma_t$. We thus have, by Ito's lemma,

$$\begin{aligned} dC^{\text{BS}}(t;K) =& \frac{\partial C^{\text{BS}}(t;K)}{\partial t} dt + \frac{\partial C^{\text{BS}}(t;K)}{\partial S} dS_t + \frac{\partial C^{\text{BS}}(t;K)}{\partial \sigma} d\sigma_t \\ &+ \frac{1}{2}\frac{\partial^2 C^{\text{BS}}(t;K)}{\partial S^2}(dS_t)^2 + \frac{1}{2}\frac{\partial^2 C^{\text{BS}}(t;K)}{\partial \sigma^2}(d\sigma_t)^2 + \frac{\partial^2 C^{\text{BS}}(t;K)}{\partial S \partial \sigma} dS_t d\sigma_t \end{aligned} \tag{19}$$

[8]A possible exception is the uncertain-parameter model of Brigo, Mercurio and Rapisarda (2004).
[9]"Continuously" typically means a daily or slightly more frequent update.

Strike	1.1750		1.2050		1.2350	
Expiry	3M	1Y	3M	1Y	3M	1Y
Hedging arguments						
Call	4.8917	8.7404	2.8409	6.7434	1.4301	5.0545
Put	0.7935	1.8740	1.7173	2.8031	3.2812	4.0401
Static replication (500 steps)						
Call	4.8963	8.7275	2.8460	6.7381	1.4325	5.0548
Pct Diff.	0.005	-0.013	0.005	-0.005	0.002	0.000
Put	0.7877	1.8690	1.7145	2.8005	3.2750	4.0396
Pct Diff.	-0.006	-0.005	-0.003	-0.003	-0.006	0.000
Static replication (3000 steps)						
Call	4.8916	8.7383	2.8433	6.7434	1.4311	5.0570
Pct Diff.	0.000	-0.002	0.002	0.000	0.001	0.002
Put	0.7885	1.8711	1.7164	2.8034	3.2785	4.0433
Pct Diff.	-0.005	-0.003	-0.001	0.000	-0.003	0.003

Table 3: Comparison of quanto option prices obtained through formulas (17) and (18).

Assuming also a Δ-hedged position and that the strikes K_i are those derived at the initial time, we immediately get

$$\begin{aligned} dC^{\rm BS}(t;K) - \sum_{i=1}^{3} x_i(t;K)\, dC^{\rm BS}(t;K_i) = &\left[\frac{\partial C^{\rm BS}(t;K)}{\partial t} - \sum_{i=1}^{3} x_i(t;K)\frac{\partial C^{\rm BS}(t;K_i)}{\partial t}\right] dt \\ &+ \left[\frac{\partial C^{\rm BS}(t;K)}{\partial \sigma} - \sum_{i=1}^{3} x_i(t;K)\frac{\partial C^{\rm BS}(t;K_i)}{\partial \sigma}\right] d\sigma_t \\ &+ \frac{1}{2}\left[\frac{\partial^2 C^{\rm BS}(t;K)}{\partial S^2} - \sum_{i=1}^{3} x_i(t;K)\frac{\partial^2 C^{\rm BS}(t;K_i)}{\partial S^2}\right] (dS_t)^2 \\ &+ \frac{1}{2}\left[\frac{\partial^2 C^{\rm BS}(t;K)}{\partial \sigma^2} - \sum_{i=1}^{3} x_i(t;K)\frac{\partial^2 C^{\rm BS}(t;K_i)}{\partial \sigma^2}\right] (d\sigma_t)^2 \\ &+ \left[\frac{\partial^2 C^{\rm BS}(t;K)}{\partial S \partial \sigma} - \sum_{i=1}^{3} x_i(t;K)\frac{\partial^2 C^{\rm BS}(t;K_i)}{\partial S \partial \sigma}\right] dS_t d\sigma_t \end{aligned} \tag{20}$$

The second, fourth and fifth term in the RHS of (20) are zero by definition of the weights x_i, whereas the third is zero due to the relation linking options' Gamma and Vega in the BS world. For the same reason, and recalling that the each option is Δ-hedged, we also have

$$\frac{\partial C^{\text{BS}}(t;K)}{\partial t} - \sum_{i=1}^{3} x_i(t;K)\frac{\partial C^{\text{BS}}(t;K_i)}{\partial t} = r^d\left[C^{\text{BS}}(t;K) - \sum_{i=1}^{3} x_i(t;K)C^{\text{BS}}(t;K_i)\right] \tag{21}$$

so that

$$dC^{\text{BS}}(t;K) - \sum_{i=1}^{3} x_i(t;K)\, dC^{\text{BS}}(t;K_i) = r^d\left[C^{\text{BS}}(t;K) - \sum_{i=1}^{3} x_i(t;K)C^{\text{BS}}(t;K_i)\right]dt \tag{22}$$

The expression in the RHS of this equation is known at time t. Therefore, the portfolio made of a long position in the call with strike K and three short positions in $x_i(t;K)$ calls with strike K_i is locally riskless at time t, in that no stochastic terms are involved in its differential.

As is well known, in the BS paradigm being long the call with strike K and short $\partial C^{\text{BS}}/\partial S$ shares of the underlying asset is equivalent to holding a locally riskless portfolio. When volatility is stochastic, and options are yet valued with the BS formula, we can still have a (locally) perfect hedge, provided that we hold suitable amounts of three different options.

One may wonder why we need three options to rule out the uncertainty due to a stochastic volatility, and not just one as typically happens when introducing a further (one-dimensional) source of randomness. The reason is twofold. First, we are not using a consistent model, but simply a valuation procedure. In fact, no two-dimensional diffusion stochastic volatility model can produce flat smiles for all maturities. Second, we are not assuming specific dynamics for the underlying and the volatility, but only a general diffusion. The three options, in fact, are also needed to rule out the model risk, since our hedging strategy is derived irrespective of the true asset and volatility dynamics (under the assumption of no jumps).

10 Conclusions

We have described a market empirical procedure to construct implied volatility curves in the FX market. We have seen that the smile construction leads to a pricing formula for any European-style contingent claim. We have then proven consistency results based on static replication and on hedging arguments.

The smile construction procedure and the related pricing formula are rather general. In fact, even though they have been developed for FX options, they can be applied in any market where three volatility quotes are available for a given maturity.

A last, unsolved issue concerns the valuation of exotic options by means of some generalization of the empirical procedure we have illustrated in this article. This is, in general, a quite complex issue to deal with, also considering that the current implied volatilities contain only information on marginal densities, which is of course not sufficient for valuing path-dependent derivatives. For exotic claims, ad-hoc procedures are usually employed.

For instance, barrier option prices can be obtained by weighing the cost difference of the "replicating" strategy by the (risk neutral) probability of not crossing the barrier before maturity. However, not only are such adjustments harder to justify theoretically than those in the plain vanilla case, but, from the practical point of view, they can even have opposite sign with respect to that implied in market prices.

Appendix A: the proofs

Proof of Proposition 3.1. Writing the system (9) in the form

$$A \begin{pmatrix} x_1(t;K) \\ x_2(t;K) \\ x_3(t;K) \end{pmatrix} = B,$$

straightforward algebra leads to

$$\begin{aligned} \det(A) &= \frac{\mathcal{V}(t;K_1)\mathcal{V}(t;K_2)\mathcal{V}(t;K_3)}{S_0\sigma^2\sqrt{T}} \big[d_2(t;K_3)d_1(t;K_2)d_2(t;K_2) + d_2(t;K_1)d_1(t;K_3)d_2(t;K_3) \\ &\quad - d_1(t;K_1)d_2(t;K_1)d_2(t;K_3) - d_1(t;K_3)d_2(t;K_3)d_2(t;K_2) \\ &\quad - d_2(t;K_1)d_1(t;K_2)d_2(t;K_2) + d_1(t;K_1)d_2(t;K_1)d_2(t;K_2)\big] \\ &= \frac{\mathcal{V}(t;K_1)\mathcal{V}(t;K_2)\mathcal{V}(t;K_3)}{S_0\sigma^5T^2} \ln\frac{K_2}{K_1} \ln\frac{K_3}{K_1} \ln\frac{K_3}{K_2} \end{aligned} \tag{23}$$

which is strictly positive since $K_1 < K_2 < K_3$. Therefore, (9) admits a unique solution and (11) follows from Cramer's rule. □

Proof of Proposition 5.1. At first order in σ, one has

$$C(K) \approx C^{\mathrm{BS}}(K) + \sum_{i=1}^{3} x_i(K)\mathcal{V}(K_i)[\sigma(K_i) - \sigma],$$

which, remembering (11) and the fact that $\sum_{i=1}^{3} x_i(K)\mathcal{V}(K_i) = \mathcal{V}(K)$, leads to

$$C(K) \approx C^{\mathrm{BS}}(K) + \mathcal{V}(K)\left[\sum_{i=1}^{3} y_i(K)\sigma(K_i) - \sigma\right],$$

where

$$\begin{aligned} y_1(K) &= \frac{\ln\frac{K_2}{K}\ \ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ \ln\frac{K_3}{K_1}} \\ y_2(K) &= \frac{\ln\frac{K}{K_1}\ \ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ \ln\frac{K_3}{K_2}} \\ y_3(K) &= \frac{\ln\frac{K}{K_1}\ \ln\frac{K}{K_2}}{\ln\frac{K_3}{K_1}\ \ln\frac{K_3}{K_2}} \end{aligned}$$

Then (13) follows from the first-order Taylor expansion

$$C(K) \approx C^{\mathrm{BS}}(K) + \mathcal{V}(K)[\sigma(K) - \sigma].$$

□

Proof of Proposition 5.2. At second order in σ, one has

$$C(K) \approx C^{\mathrm{BS}}(K) + \sum_{i=1}^{3} x_i(K) \left[\mathcal{V}(K_i)(\sigma(K_i) - \sigma) + \frac{1}{2}\frac{\partial^2 C^{\mathrm{BS}}}{\partial^2 \sigma}(K_i)(\sigma(K_i) - \sigma)^2 \right].$$

Analogously,

$$C(K) - C^{\mathrm{BS}}(K) \approx \mathcal{V}(K)(\sigma(K) - \sigma) + \frac{1}{2}\frac{\partial^2 C^{\mathrm{BS}}}{\partial^2 \sigma}(K)(\sigma(K) - \sigma)^2 \Big],$$

so that we can write

$$\begin{aligned} &\mathcal{V}(K)(\sigma(K) - \sigma) + \frac{1}{2}\frac{\partial^2 C^{\mathrm{BS}}}{\partial^2 \sigma}(K)(\sigma(K) - \sigma)^2 \\ &\approx \sum_{i=1}^{3} x_i(K) \left[\mathcal{V}(K_i)(\sigma(K_i) - \sigma) + \frac{1}{2}\frac{\partial^2 C^{\mathrm{BS}}}{\partial^2 \sigma}(K_i)(\sigma(K_i) - \sigma)^2 \right]. \end{aligned}$$

Solving this algebraic second-order equation in $\sigma(K)$ then leads to (14). □

Proof of Proposition 6.1. The equality (16) holds if and only if

$$\sum_{j=1}^{3} x_j(K; \mathcal{H})[C^{\mathcal{H}}(H_j) - C^{\mathrm{BS}}(H_j)] = \sum_{i=1}^{3} x_i(K; \mathcal{K})[C(K_i) - C^{\mathrm{BS}}(K_i)]$$

Using (15) and rearranging terms, the left hand side can be written as

$$\begin{aligned} \sum_{j=1}^{3} x_j(K; \mathcal{H})[C^{\mathcal{H}}(H_j) - C^{\mathrm{BS}}(H_j)] &= \sum_{j=1}^{3} x_j(K; \mathcal{H}) \sum_{i=1}^{3} x_i(H_j; \mathcal{K})[C(K_i) - C^{\mathrm{BS}}(K_i)] \\ &= \sum_{i=1}^{3} \left[\sum_{j=1}^{3} x_j(K; \mathcal{H}) x_i(H_j; \mathcal{K}) \right] [C(K_i) - C^{\mathrm{BS}}(K_i)] \end{aligned}$$

which equals the right hand side of the above equality since, for each strike K and $j = 1, 2, 3$,

$$x_i(K; \mathcal{K}) = \sum_{j=1}^{3} x_j(K; \mathcal{H}) x_i(H_j; \mathcal{K}) \tag{24}$$

following from a tedious, but straightforward, application of the formula (11) for the weights. □

Proof of Proposition 7.1. For each operator $\mathcal{L} \in \{\frac{\partial}{\partial\sigma}, \frac{\partial^2}{\partial^2\sigma}, \frac{\partial^2}{\partial\sigma\partial S_0}\}$ we have

$$\begin{aligned}\mathcal{L}V^{\text{BS}} &= \mathcal{L}\left[e^{-r^d T}h(0) + S_0\, e^{-r^f T}h'(0) + \int_0^{+\infty} h''(K)C^{\text{BS}}(K)\,dK\right] \\ &= \int_0^{+\infty} h''(K)\mathcal{L}C^{\text{BS}}(K)\,dK\end{aligned}$$

which, by definition of the weights $x_i(K)$, becomes

$$\begin{aligned}\mathcal{L}V^{\text{BS}} &= \int_0^{+\infty} h''(K)\sum_{i=1}^{3} x_i(K)\,\mathcal{L}C^{\text{BS}}(K_i)\,dK \\ &= \sum_{i=1}^{3}\int_0^{+\infty} h''(K)x_i(K)\,\mathcal{L}C^{\text{BS}}(K_i)\,dK \\ &= \sum_{i=1}^{3}\left[\int_0^{+\infty} h''(K)x_i(K)\,dK\right]\mathcal{L}C^{\text{BS}}(K_i)\end{aligned}$$

By the uniqueness of the weights x_i^h we thus have

$$x_i^h = \int_0^{+\infty} h''(K)x_i(K)\,dK, \quad i = 1, 2, 3$$

Substituting into (17), we get

$$\begin{aligned}\bar{V} &= V^{\text{BS}} + \sum_{i=1}^{3}\left[\int_0^{+\infty} h''(K)x_i(K)\,dK\right][C(K_i) - C^{\text{BS}}(K_i)] \\ &= V^{\text{BS}} + \int_0^{+\infty} h''(K)\sum_{i=1}^{3} x_i(K)[C(K_i) - C^{\text{BS}}(K_i)]\,dK \\ &= V^{\text{BS}} + \int_0^{+\infty} h''(K)[C(K) - C^{\text{BS}}(K)]\,dK \\ &= V^{\text{BS}} + [V - V^{\text{BS}}] = V\end{aligned}$$

□

11 Appendix B: the implied risk-neutral density

The VV price (12) is defined without introducing specific assumptions on the distribution of the underlying asset. However, the knowledge of option prices for every possible strike implicitly determines a unique risk-neutral density that is consistent with them. In fact,

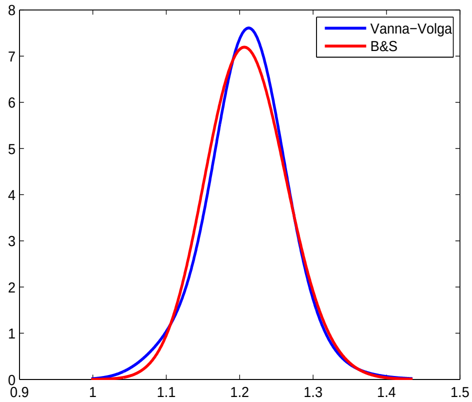

Figure 4: Vanna-Volga risk-neutral density compared with the lognormal one coming from the BS model with ATM volatility.

by the general result of Breeden and Litzenberger (1978), the risk-neutral density p_T of the exchange rate S_T can be obtained by differentiating twice the option price (12):

$$p_T(K) = e^{r^d T}\frac{\partial^2 C}{\partial K^2}(K) = e^{r^d T}\frac{\partial^2 C^{\rm BS}}{\partial K^2}(K) + e^{r^d T}\sum_{i\in\{1,3\}}\frac{\partial^2 x_i}{\partial K^2}(K)[C^{\rm MKT}(K_i) - C^{\rm BS}(K_i)]. \quad (25)$$

The first term in the RHS is the lognormal density $p_T^{\rm BS}$ associated with the geometric Brownian motion with drift rate $r^d - r^f$ and volatility σ. The second term, which is the deviation from lognormality induced by the VV smile, is more involved and can be calculated by differentiating twice the weights (11). We obtain:

$$\begin{aligned}\frac{\partial^2 x_1}{\partial K^2}(K) =& \frac{\mathcal{V}(K)}{K^2\sigma^2 T\mathcal{V}(K_1)\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_1}}\bigg[\Big(d_1(K)^2 - \sigma\sqrt{T}d_1(K) - 1\Big)\ln\frac{K_2}{K}\ln\frac{K_3}{K}\\ &- 2\sigma\sqrt{T}d_1(K)\ln\frac{K_2K_3}{K^2} + \sigma^2 T\Big(\ln\frac{K_2K_3}{K^2} + 2\Big)\bigg]\\ \frac{\partial^2 x_3}{\partial K^2}(K) =& \frac{\mathcal{V}(K)}{K^2\sigma^2 T\mathcal{V}(K_3)\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}}\bigg[\Big(d_1(K)^2 - \sigma\sqrt{T}d_1(K) - 1\Big)\ln\frac{K_2}{K}\ln\frac{K_1}{K}\\ &- 2\sigma\sqrt{T}d_1(K)\ln\frac{K_1K_2}{K^2} + \sigma^2 T\Big(\ln\frac{K_1K_2}{K^2} + 2\Big)\bigg].\end{aligned}$$

A plot of the risk-neutral density associated to (12) is shown in Figure 4, where it is compared with the corresponding lognormal density $p_T^{\rm BS}$.

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