No-arbitrage conditions for cash-settled swaptions

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Abstract

In this note, we derive no-arbitrage conditions that must be satisfied by the pricing function of cash-settled swaptions. The specific examples of a flat implied volatility and of a smile generated by the SABR functional form will be analyzed.

1 Introduction and purpose of the article

Cash-settled swaptions are the most actively traded swaptions in the Euro market. Their payoff is obtained by replacing the classical annuity term with a single-factor one, where discounting is based on a unique interest rate, namely the underlying swap rate set at the option's maturity.

The market typically uses a Black-like formula for pricing cash-settled swaptions. However, such a formula can not be immediately justified in terms of a market model, since the cash-settled annuity term can not be regarded as a proper numeraire.

In this note, we derive necessary and sufficient conditions for a pricing function for cash-settled swaptions to be arbitrage free. In particular, we will analyze possible faults of the market formula when the whole smile for a given swaption is considered. Specific examples based on a flat smile and on the SABR functional form will be provided.

Finally, in the appendix, we will identify a strategy leading to an arbitrage in case the above conditions are not met.

2 Definitions

Let us fix a maturity T_a and a set of times $\mathcal{T}_{a,b} := \{T_{a+1}, \dots, T_b\}$, with associated year fractions all equal to $\tau > 0$. The forward swap rate at time t for payments in $\mathcal{T}_{a,b}$ is defined by

$$S_{a,b}(t) = \frac{P(t,T_a) - P(t,T_b)}{\tau \sum_{j=a+1}^{b} P(t,T_j)},$$

where $P(t,T)$ denotes the time-t discount factor for maturity T.

A cash-settled swaption is an option paying out at time T_a

$$\big[\omega\big(S_{a,b}(T_a) - K\big)\big]^+ G_{a,b}(S_{a,b}(T_a)), \tag{1}$$

where K is the option's strike, $\omega = 1$ for a payer and $\omega = -1$ for a receiver, and

$$G_{a,b}(S) := \sum_{j=1}^{b-a} \frac{\tau}{(1+\tau S)^j} = \begin{cases} \frac{1}{S}\Big[1 - \frac{1}{(1+\tau S)^{b-a}}\Big] & S > 0 \\ \tau(b-a) & S = 0 \end{cases}$$

is the cash-settled annuity term.

Denoting by Q^{T_a} the T_a-forward measure, and by E^{T_a} the related expectation, the no-arbitrage price at time zero of the derivative (1) is given by

$$\mathbf{CSS}(K, T_a; \mathcal{T}_{a,b}, \omega) = P(0,T_a) E^{T_a}\big\{\big[\omega\big(S_{a,b}(T_a) - K\big)\big]^+ G_{a,b}(S_{a,b}(T_a))\big\}. \tag{2}$$

Contrary to physically-settled swaptions paying out, at time T_a,

$$\big[\omega\big(S_{a,b}(T_a) - K\big)\big]^+ \sum_{j=a+1}^{b} \tau P(T_a, T_j),$$

where the annuity term $\sum_{j=a+1}^{b} \tau P(\cdot, T_j)$ is the value of a portfolio of tradable assets, the cash-settled annuity $G_{a,b}(S_{a,b}(\cdot))$ is not a proper numeraire, which makes it impossible to value (1) analytically by means of an analogue of the swap market model.

Nevertheless, it is market practice to value (2) with the following Black-like formula:

$$\mathbf{CSS}^{\text{MKT}}(K, T_a; \mathcal{T}_{a,b}, \omega) = P(0,T_a) c_{a,b}^{\text{BS}}(K;\omega) G_{a,b}(S_{a,b}(0)), \tag{3}$$

where

$$\begin{aligned} c_{a,b}^{\text{BS}}(K;\omega) &:= \text{Bl}\big(K, S_{a,b}(0), \sigma_{a,b}^{\text{MKT}}(K)\sqrt{T_a}, \omega\big), \\ \text{Bl}(K,S,v,\omega) &:= \omega S \Phi\bigg(\omega \frac{\ln(S/K) + v^2/2}{v}\bigg) - \omega K \Phi\bigg(\omega \frac{\ln(S/K) - v^2/2}{v}\bigg), \end{aligned}$$

and where $\sigma_{a,b}^{\text{MKT}}(K)$ is the market implied volatility for the swaption struck at K and Φ denotes the standard normal cumulative distribution function.

However, the question remains of whether (3) is an arbitrage-free formula. In the following, we will try to address this issue starting from a general pricing function.

3 Conditions for the absence of arbitrage

Let us assume that the Q^{T_a}-distribution of $S_{a,b}(T_a)$ is absolutely continuous and denote by $p_{a,b}(\cdot)$ the associated density (with finite mean). Let us also assume that cash-settled swaptions, with maturity T_a and underlying tenor structure $\mathcal{T}_{a,b}$, are priced with a function $\pi_{a,b}(K;\omega)$, which is positive and twice differentiable in K for given ω. Clearly, by our assumptions, we must have (assuming positive rates)

$$\pi_{a,b}(K;\omega) = P(0,T_a)\int_0^{+\infty}[\omega(x-K)]^+ G_{a,b}(x)p_{a,b}(x)\,dx, \tag{4}$$

or, equivalently,

$$\begin{aligned}\pi_{a,b}(K;1) &= P(0,T_a)\int_K^{+\infty}(x-K)G_{a,b}(x)p_{a,b}(x)\,dx,\\ \pi_{a,b}(K;-1) &= P(0,T_a)\int_0^{K}(K-x)G_{a,b}(x)p_{a,b}(x)\,dx.\end{aligned} \tag{5}$$

The given density $p_{a,b}$ uniquely identifies the price function $\pi_{a,b}$ through (5). The reverse is also true. In fact, following Breeden and Litzenberger (1978), and differentiating twice both equalities in (5), we obtain:

$$\pi''_{a,b}(K;1) = \pi''_{a,b}(K;-1) = P(0,T_a)G_{a,b}(K)p_{a,b}(K), \tag{6}$$

namely

$$p_{a,b}(K) = \frac{\pi''_{a,b}(K;1)}{P(0,T_a)G_{a,b}(K)}, \tag{7}$$

where $'$ denotes derivative with respect to K. Therefore, if we are given an arbitrage-free price function $\pi_{a,b}$, we can uniquely identify the density $p_{a,b}$ through (7).

Equation (7) also helps us characterize the price functions for cash-settled swaptions that are arbitrage free in the sense that (4) holds, for each K, for some density function $p_{a,b}$. Precisely, we have the following.

Proposition 3.1. *A given (twice-differentiable) function $\pi_{a,b}$ is an arbitrage-free price function for cash-settled swaptions if and only if*

i) $\pi''_{a,b}(K;1) = \pi''_{a,b}(K;-1)$ *for each* K*;*

ii) $g : K \mapsto \frac{\pi''_{a,b}(K;1)}{P(0,T_a)G_{a,b}(K)}$ *is a density function, namely:*

– *the second derivative of* $\pi_{a,b}$ *is positive:* $\pi''_{a,b}(K;1) \geq 0$;

– *the function* g *is normalized to unity:*

$$\int_0^{+\infty} \frac{\pi''_{a,b}(x;1)}{P(0,T_a)G_{a,b}(x)}\,dx = 1 \;\Leftrightarrow\; \int_0^{+\infty} \frac{\pi''_{a,b}(x;1)}{G_{a,b}(x)}\,dx = P(0,T_a); \tag{8}$$

iii) the following boundary conditions apply:

$$\begin{aligned}
&\lim_{K\to+\infty} \pi_{a,b}(K;1) = 0,\\
&\lim_{K\to+\infty} K\pi'_{a,b}(K;1) = 0,\\
&\pi_{a,b}(0;-1) = 0,\\
&\pi'_{a,b}(0;-1) = 0.
\end{aligned} \tag{9}$$

Proof. Conditions i) and ii) follow from (6) and (7), respectively. Conditions iii) follow from imposing that, for each K,

$$\begin{aligned}
\pi_{a,b}(K;1) &= \int_K^{+\infty} (x-K)\pi''_{a,b}(x;1)\,dx,\\
\pi_{a,b}(K;-1) &= \int_0^{K} (K-x)\pi''_{a,b}(x;-1)\,dx.
\end{aligned}$$

□

Conditions i) to iii) are rather standard and, mutatis mutandis, typical of any pricing function for European calls and puts. However, in the case of cash-settled swaptions (*i.e.* when $G_{a,b}$ is not identically equal to one), the normalization property (8) is not automatically granted and needs to be verified on a case-by-case basis.

Remark 3.2. *For standard European calls and puts, it is well known how to construct arbitrage opportunities in case (at least) one of conditions i) to iii) is not fulfilled. For instance, if the second derivative* $\pi''_{a,b}(K;1)$ *is strictly negative, we can find a butterfly swaption spread with positive payoff but (strictly) negative price.*[1] *In fact,* $\pi''_{a,b}(K;1) < 0$ *for some* K *implies the existence of* $\epsilon > 0$ *such that* $[\pi_{a,b}(K+\epsilon;1) - 2\pi_{a,b}(K;1) + \pi_{a,b}(K-\epsilon;1)]/(\epsilon^2) < 0$ *so that two short positions in the payer swaption with strike* K *plus long positions in the payer swaptions with strikes* $K-\epsilon$ *and* $K+\epsilon$ *would yield an arbitrage opportunity (payoff* ≥ 0 *but price* < 0*).*

When the normalization property (8) is not satisfied, the construction of an arbitrage is less straightforward and is explained in the appendix.

[1] See for instance Carr and Madan (2005).

4 Is the market formula arbitrage free?

With the above conditions at hand, we can now proceed to verify whether the market formula (3) is indeed arbitrage free. To this end, we will consider two specific cases. The first is based on a flat smile. The second on modelling implied volatilities with the SABR functional form of Hagan et al. (2002), which is quite popular in the swaption market, see for instance Mercurio and Pallavicini (2005, 2006).

The first case we analyze is that where implied volatilities are constant for each K, $\sigma_{a,b}^{\mathrm{MKT}}(K) = \bar{\sigma}_{a,b}$:

$$\pi_{a,b}(K,\omega) = P(0,T_a)\mathrm{Bl}\big(K, S_{a,b}(0), \bar{\sigma}_{a,b}\sqrt{T_a}, \omega\big) G_{a,b}(S_{a,b}(0)). \tag{10}$$

Being this pricing function equal to Black's formula times a constant, conditions *i*), the first of *ii*) and *iii*) of Proposition 3.1 are immediately satisfied. We are left with the normalization (8), which becomes

$$G_{a,b}(S_{a,b}(0)) \int_0^{+\infty} \frac{\frac{d^2}{dx^2}\mathrm{Bl}\big(x, S_{a,b}(0), \bar{\sigma}_{a,b}\sqrt{T_a}, 1\big)}{G_{a,b}(x)}\, dx = 1. \tag{11}$$

The integral in the left-hand-side is the expected value of the random variable $1/G_{a,b}(X)$, where X is lognormally distributed with mean $S_{a,b}(0)$ and second moment $S_{a,b}^2(0)\exp\{\bar{\sigma}_{a,b}^2 T_a\}$. Such a value clearly depends on $\bar{\sigma}_{a,b}$. Therefore, there seems to be no a priori reason why condition (11) should hold for the given volatility $\bar{\sigma}_{a,b}$. In fact, setting $n := b - a$, we have the following.

Proposition 4.1. *The normalization (11) holds true if and only if $n = 1$ or $\bar{\sigma}_{a,b} = 0$.*

Proof. The case $n = 1$ is trivial. In this case, in fact, a cash-settled swaption becomes a caplet and (10) reduces to the corresponding Black formula.

When $n > 1$, we notice that the function $1/G_{a,b}(x)$ is (strictly) convex since:

$$\frac{d^2}{dx^2}\frac{1}{G_{a,b}(x)} = n\tau \frac{(1+\tau x)^n(n\tau x - \tau x - 2) + n\tau x + \tau x + 2}{(1+\tau x)^{2n+2}\big(1-(1+\tau x)^{-n}\big)^3} > 0 \quad \text{for } x > 0,$$

as we get by expanding the power in the numerator in the right-hand-side. Therefore, by the positivity of a convex payoff's Vega in a Black and Scholes world, the integral in (11) is a strictly increasing function in $\bar{\sigma}_{a,b}$, whose infimum is then attained for $\bar{\sigma}_{a,b} = 0$. Since the lognormal density in (11) collapses to a Dirac delta concentrated at $S_{a,b}(0)$ when the volatility tends to zero, the limit value for the integral is $1/G_{a,b}(S_{a,b}(0))$. As a consequence, (11) is fulfilled in the limit case $\bar{\sigma}_{a,b} = 0$ but violated for every $\bar{\sigma}_{a,b} > 0$ (when $n > 1$). □

The second case we consider is that where implied volatilities are modelled with the SABR functional form:

$$\begin{aligned}
\sigma_{a,b}^{\text{MKT}}(K) = \sigma_{a,b}^{\text{SABR}}(K) &:= \frac{\alpha}{(S_{a,b}(0)K)^{\frac{1-\beta}{2}}\left[1+\frac{(1-\beta)^2}{24}\ln^2\left(\frac{S_{a,b}(0)}{K}\right)+\frac{(1-\beta)^4}{1920}\ln^4\left(\frac{S_{a,b}(0)}{K}\right)\right]}\frac{z}{x(z)} \\
&\cdot\left\{1+\left[\frac{(1-\beta)^2\alpha^2}{24(S_{a,b}(0)K)^{1-\beta}}+\frac{\rho\beta\epsilon\alpha}{4(S_{a,b}(0)K)^{\frac{1-\beta}{2}}}+\epsilon^2\frac{2-3\rho^2}{24}\right]T_a\right\}, \\
z &:= \frac{\epsilon}{\alpha}(S_{a,b}(0)K)^{\frac{1-\beta}{2}}\ln\left(\frac{S_{a,b}(0)}{K}\right), \\
x(z) &:= \ln\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\},
\end{aligned} \tag{12}$$

where $\alpha > 0$, $\beta \in [0,1]$, $\epsilon > 0$ and $\rho \in (-1,1)$ are the model parameters,[2] leading to the pricing function:

$$\pi_{a,b}(K,\omega) = P(0,T_a)\text{Bl}\big(K, S_{a,b}(0), \sigma_{a,b}^{\text{SABR}}(K)\sqrt{T_a}, \omega\big)G_{a,b}(S_{a,b}(0)). \tag{13}$$

It is well known that the SABR option prices are not always arbitrage free since

$$\lim_{K\to 0^+}\frac{d}{dK}\text{Bl}\big(K, S_{a,b}(0), \sigma_{a,b}^{\text{SABR}}(K)\sqrt{T_a}, 1\big) = 0, \tag{14}$$

implying that

$$\frac{d^2}{dK^2}\text{Bl}\big(K, S_{a,b}(0), \sigma_{a,b}^{\text{SABR}}(K)\sqrt{T_a}, 1\big) < 0. \tag{15}$$

for small strikes. Accordingly, also the cash-settled swaption price (13) fails to fulfil the positivity condition for its second derivative, and hence we can already conclude that the price function (13) is not arbitrage free (an arbitrage can be built with a suitable butterfly).

But the situation is even worse. In fact, due to (14) and (15), nor is the normalization property

$$G_{a,b}(S_{a,b}(0))\int_0^{+\infty}\frac{\frac{d^2}{dx^2}\text{Bl}\big(x, S_{a,b}(0), \sigma_{a,b}^{\text{SABR}}(x)\sqrt{T_a}, 1\big)}{G_{a,b}(x)}\,dx = 1 \tag{16}$$

satisfied. Again, this is direct consequence of the functional form (12) and not of its specific use in the pricing of cash-settled swaption.

[2]Precisely, α is the initial value of the underlying asset's volatility, β the constant-elasticity-of-variance parameter in the asset dynamics, $\epsilon > 0$ the volatility of volatility and ρ the instantaneous correlation between the asset and its volatility.

The normalization property (8) in the SABR case is violated (in non-trivial situations) even when we replace the limit (14) with the no-arbitrage one (which is -1) and integrate by parts obtaining:

$$G_{a,b}(S_{a,b}(0))\left[f_{a,b}(0)+f'_{a,b}(0)S_{a,b}(0)+\int_0^{+\infty} f''_{a,b}(x)\mathrm{Bl}\big(x,S_{a,b}(0),\sigma_{a,b}^{\mathrm{SABR}}(x)\sqrt{T_a},1\big)\,dx\right]=1, \tag{17}$$

where we set $f_{a,b}(x) := G_{a,b}(x)$.[3]

Similarly to the flat-smile case, there is no reason why (17) should be satisfied for each parameter quadruplet $(\alpha, \beta, \epsilon, \rho)$.[4] In fact, if the SABR implied volatility (12) has a positive (partial) derivative with respect to α, which holds true for realistic model parameters, we have the following.

Proposition 4.2. *Assuming that the implied volatility (12) is an increasing function of α, (17) holds true if and only if $\alpha = 0$ or $n = 1$.*

Proof. The case $n = 1$ is trivial. In this case, in fact, $f_{a,b}(x) = x + 1/\tau$, and hence $f''_{a,b} \equiv 0$, $f_{a,b}(0) = 1/\tau$ and $f'_{a,b}(0) = 1$.

Instead, when $n > 1$ and $\frac{\partial}{\partial\alpha}\sigma_{a,b}^{\mathrm{SABR}} > 0$, the positivity of a call's Vega in the Black and Scholes world implies that the left-hand-side of (17) is an increasing function of α, whose infimum is then attained for $\alpha = 0$. The value of the left-hand-side in this limit case is indeed one. □

Finally, let us consider a numerical example supporting the above conclusions in the SABR case. Assume we are given some market volatilities for an underlying forward swap rate with 20 year expiry and 10 year tenor, as those reported in Figure 1. Setting $\beta = 0.6$, we calibrate the functional form (12) to such volatilities obtaining: $\alpha = 0.033596$, $\epsilon = 0.242041$ and $\rho = -0.266455$. With these values for the SABR parameters we then calculate the left-hand-side of (17). The value we get is 1.005. A graphical evidence of the statement of Proposition 4.2 is given in Figure 2, where we plot the evolution of the left-hand-side of (17) for different values of α.

[3]The main reason for resorting to an integration by parts is because the integral in (17) is easier to calculate than that in (16).

[4]Notice also that (12) contains the constant-volatility example as a limit case.

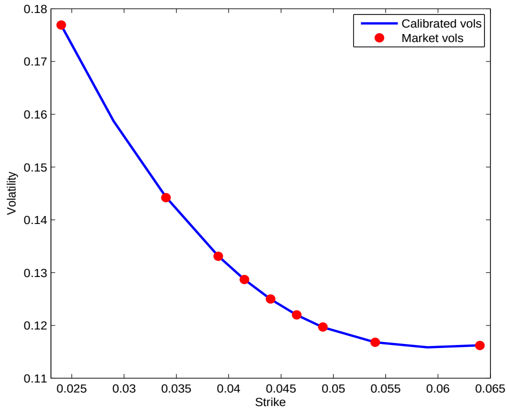

Figure 1: SABR calibrated volatilities compared with market ones for 20x10 swaptions, with $S_{a,b}(0) = 0.044$.

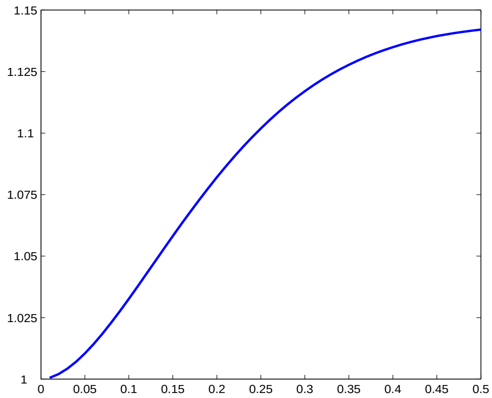

Figure 2: Value of the left-hand-side of (17) for $\beta = 0.6$, $\epsilon = 0.242041$, $\rho = -0.266455$ and different values of α.

5 Conclusions

References

[1] Breeden, D.T. and Litzenberger, R.H. (1978) Prices of State-Contingent Claims Implicit in Option Prices. *Journal of Business*, 51, 621-651.

[2] Carr, P. and Madan, D.B. (2005) A note on sufficient conditions for no arbitrage. *Finance Research Letters* 2, 125-130.

[3] Hagan, P.S., Kumar, D., Lesniewski, A.S., Woodward, D.E. (2002) Managing Smile Risk. *Wilmott magazine*, September, 84-108.

[4] Mercurio, F. and Pallavicini, A. (2005) Swaption skews and convexity adjustments. Working paper. Available on line at: `http://www.fabiomercurio.it/sabrcms.pdf`

[5] Mercurio F. and Pallavicini A. (2006) Smiling at Convexity. *Risk*, August, 64-69.

Appendix: Constructing an arbitrage when (8) is not fulfilled

Let us suppose that, besides cash-settled swaptions, we can also trade CMS caplets, floorlets and swaplets, paying respectively $[S_{a,b}(T_a)-K]^+$, $[K-S_{a,b}(T_a)]^+$ and $[S_{a,b}(T_a)-K]$ at time T_a. Assuming that cash-settled swaptions are priced with a function $\pi_{a,b}$, remembering (7), the prices of CMS options are given by:

$$\mathbf{CMSCplt}(S_{a,b}, K) = P(0,T_a)E^{T_a}[(S_{a,b}(T_a)-K)^+] = \int_K^{+\infty} (x-K)\frac{\pi''_{a,b}(x;1)}{G_{a,b}(x)}\,dx \tag{18}$$

and

$$\mathbf{CMSFllt}(S_{a,b}, K) = P(0,T_a)E^{T_a}[(K-S_{a,b}(T_a))^+] = \int_0^{K} (K-x)\frac{\pi''_{a,b}(x;1)}{G_{a,b}(x)}\,dx. \tag{19}$$

Since

$$[S_{a,b}(T_a)-K]^+ - [K-S_{a,b}(T_a)]^+ = S_{a,b}(T_a)-K,$$

to avoid arbitrage one should have

$$\mathbf{CMSCplt}(S_{a,b}, K) - \mathbf{CMSFllt}(S_{a,b}, K) = \mathbf{CMSCplt}(S_{a,b}, 0) - KP(0,T_a). \tag{20}$$

Plugging (18) and (19) into (20), one gets

$$\int_K^{+\infty} (x-K)\frac{\pi''_{a,b}(x;1)}{G_{a,b}(x)}\,dx - \int_0^K (K-x)\frac{\pi''_{a,b}(x;1)}{G_{a,b}(x)}\,dx = \int_0^{+\infty} x\frac{\pi''_{a,b}(x;1)}{G_{a,b}(x)}\,dx - KP(0,T_a),$$

which holds true if and only if

$$\int_0^{+\infty} \frac{\pi''_{a,b}(x;1)}{G_{a,b}(x)}\,dx = P(0,T_a),$$

namely if and only if (8) is fulfilled.

Therefore, assuming, for instance, that

$$\int_0^{+\infty} \frac{\pi''_{a,b}(x;1)}{G_{a,b}(x)}\,dx < P(0,T_a), \tag{21}$$

one would have

$$\mathbf{CMSCplt}(S_{a,b},0) - \mathbf{CMSCplt}(S_{a,b},K) + \mathbf{CMSFllt}(S_{a,b},K) < KP(0,T_a), \tag{22}$$

so that being long the swaplet and the floorlet and short the caplet and the T_a-maturity bond would yield an arbitrage opportunity (zero payoff but price < 0).

The case with the " $>$ " sign in (21) is perfectly analogous.