Pricing the smile in a forward LIBOR market model

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Abstract

We introduce two general classes of analytically-tractable diffusions for modeling forward LIBOR rates under their canonical measure.

The first class is based on the assumption of forward-rate densities given by the mixture of known basic densities. We consider two fundamental examples: i) a mixture of lognormal densities, and ii) a mixture of densities associated to "hyperbolic-sine" processes. We derive explicit dynamics, prove existence and uniqueness results for the solution to the related SDEs and obtain closed-form formulas for caps prices.

The second class is based on assuming a smooth functional dependence, at expiry, between a forward rate and an associated Brownian motion. This class is highly tractable: it implies explicit dynamics, known marginal and transition densities and explicit caplet prices at any time. As an example, we analyze the dynamics given by a linear combination of geometric Brownian motions (GBM) with perfectly correlated (decorrelated) returns. We finally construct a specific model in the second class that reproduces exactly the market caplet volatilities given in input.

Examples of the implied-volatility curves produced by the considered models are also shown.

1 Introduction

The market models are nowadays the most popular interest-rate models both among academics and among practitioners. Their success is mainly due to the possibility of reproducing exactly the market Black formulas for either caplets or swaptions. Two are the type of market models one can consider: the forward LIBOR model (FLM) and the swap market model, respectively leading to Black's formulas for caps and for swaptions, when expressed in their *lognormal formulation*.

Given its better tractability and the payoffs nature one commonly encounters in practice, the FLM turns out to be the most convenient choice in many situations.

However, the FLM property of exactly retrieving the market Black formula only applies to standard at-the-money (ATM) caplets, meaning that a tangible mispricing may be produced

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for away-from-the-money options. In fact, in real cap markets, the implied volatility curves are typically skew- or smile-shaped.[1]

In this paper, we try and address the issue of defining FLM dynamics that are alternative to the classical lognormal ones and are capable of retrieving implied volatility structures as typically observed in the market.

Many researchers have tried to address the problem of a good, possibly exact, fitting of market option data. We now briefly review the major approaches proposed in the existing literature. These approaches, though mainly developed in specific contexts, can also be applied to the case of a general underlying asset, and to a forward rate in particular.

A first approach is based on assuming an *alternative explicit dynamics* for the asset-price process that immediately leads to volatility smiles or skews. In general this approach does not provide sufficient flexibility to properly calibrate the whole volatility surface. An example is the constant-elasticity-of-variance (CEV) process analyzed by Cox (1975) and Cox and Ross (1976), with the related application to the FLM being considered by Andersen and Andreasen (2000). A general class of processes is due to Carr et al. (1999). The first class of models we propose also fall into this *alternative explicit dynamics* category, and while it adds flexibility with respect to the previous known examples, it does not completely solve the flexibility issue.

A second approach is based on the assumption of a *continuum of traded strikes* and goes back to Breeden and Litzenberger (1978). Successive developments are due, among all, to Dupire (1994, 1997) and Derman and Kani (1994, 1998) who derive an explicit expression for the Black-Scholes volatility as a function of strike and maturity. This approach has the major drawback that one needs to smoothly interpolate option prices between consecutive strikes in order to be able to differentiate them twice with respect to the strike. Explicit expressions for the risk-neutral stock price dynamics are also derived by Avellaneda et al. (1997) by minimizing the relative entropy to a prior distribution.

Another approach, pioneered by Rubinstein (1994), consists of finding the risk-neutral probabilities in a binomial/trinomial model for the asset price that lead to a best fitting of market option prices due to some smoothness criterion. We refer to this approach as to the *lattice* approach. Further examples are in Jackwerth and Rubinstein (1996) and Britten-Jones and Neuberger (2000).

A further approach is given by what we may refer to as *incomplete-market* approach. It includes stochastic-volatility models, such as those of Hull and White (1987), Heston (1993) and Tompkins (2000a, 2000b), and jump-diffusion models, such as those of Merton (1976) or Prigent, Renault and Scaillet (2001). In the context of the FLM, we must mention the recent works of Balland, P. and Hughston L.P. (2000), who allow for a broad class of LIBOR rate distributions, and Rebonato (2001), who considers stochastic volatility.

A last approach is based on the so called *market model for implied volatility*. The first examples are in Schönbucher (1999) and Ledoit and Santa Clara (1998). A recent application to the FLM case is due to Brace et al. (2001).

In general the problem of finding a risk-neutral distribution that consistently prices all quoted options is largely undetermined. A possible solution is given by assuming a particular *parametric risk-neutral distribution* depending on several, possibly time-dependent, parameters and then use such parameters for the volatility calibration. By applying an approach

[1]The term "skew" is used to indicate those structures where low-strikes implied volatilities are higher than high-strikes implied volatilities. The term "smile" is used instead to denote those structures with a minimum value around the current value of the underlying forward rate.

similar to that of Dupire (1994, 1997), we address this question and find a first class of dynamics leading to parametric risk-neutral distributions that are flexible enough for practical purposes. The resulting processes combine therefore the *parametric risk-neutral distribution* approach with the *alternative dynamics approach*, providing explicit dynamics that lead to flexible parametric risk-neutral densities. Under the lognormal-mixture assumption, we basically apply the results of Brigo and Mercurio (2000, 2001a, 2001b, 2002b) to the FLM case.

The major challenge that our models in this class are able to face is the introduction of a (forward-measure) distribution that leads i) to *analytical formulas* for caplets, and hence caps, so that the calibration to market data and the computation of Greeks can be extremely rapid, ii) to *explicit forward LIBOR dynamics*, so that exotic claims can be priced through a Monte Carlo simulation.

We then introduce a second general class, which is based on assuming a smooth functional dependence, at expiry, between a forward rate and an associated Brownian motion. This class is even more tractable than the previous one: it implies explicit dynamics, known marginal and transition densities and explicit option prices at any time. As an example, we consider the dynamics given by a linear combination of geometric Brownian motions with perfectly correlated (decorrelated) returns. We also construct a model in this second class that is capable of exactly calibrating the market volatilities given in input.

The paper is structured as follows. Section 2 reviews the smile problem in the context of the FLM. Section 3 explains the shifted-lognormal and the CEV models as applied to the FLM. Section 4 proposes a general class of asset-price models based on marginal densities that are given by the mixture of some basic densities. Section 5 considers the particular case of a mixture of lognormal densities and derives closed form formulas for option prices and analytical approximations for the implied volatility function. Section 6 introduces the asset-price model that is obtained by shifting the previous lognormal-mixture dynamics and investigate its analytical tractability. Section 7 proposes two other examples: the first is still based on lognormal densities, but it allows for different means; the second is instead based on basic dynamics of hyperbolic-sine type. Section 8 introduces the second general class, which is based on assuming a smooth functional dependence, at expiry, between a forward rate and an associated Brownian motion. Section 9 considers a specific example of calibration to real market caps data. Section 10 concludes the paper.

2 A Mini-tour on the Smile Problem

It is well known that Black's formula for caplets is the standard in the cap market. This formula is consistent with the lognormal FLM, in that it comes as the expected value of the discounted caplet payoff under the related forward measure when the forward-rate dynamics is given by the FLM.

To fix ideas, let us consider the time-0 price of a T_2-maturity caplet resetting at time T_1 ($0 < T_1 < T_2$) with strike K and a notional amount of 1. Let τ denote the year fraction between T_1 and T_2. Such a contract pays out, at time T_2, the amount

$$\tau(F(T_1; T_1, T_2) - K)^+,$$

where in general $F(t; S, T)$ denotes the forward LIBOR rate, at time t, from expiry S to maturity T, i.e.

$$F(t;S,T) = \frac{1}{\tau(S,T)}\left[\frac{P(t,S)}{P(t,T)} - 1\right],$$

with $\tau(S,T)$ the year fraction from S to T, and $P(t,s)$ the discount factor at time t for maturity s.

Usual no-arbitrage arguments imply that the value at time 0 of the contract is

$$P(0,T_2)\tau E^2[(F(T_1;T_1,T_2) - K)^+],$$

with E^2 denoting expectation with respect to the T_2-forward measure Q^2.

Assume that the Q^2-dynamics for the above F is that of the lognormal FLM

$$dF(t;T_1,T_2) = \sigma_2(t)F(t;T_1,T_2)\,dW_t\,, \tag{1}$$

where σ_2 is a deterministic function of time. Lognormality of F's density at time T_1 implies that the above expectation results in the following Black formula:

$$\mathbf{Cpl}^{\text{Black}}(0,T_1,T_2,K) = P(0,T_2)\tau \text{Bl}(K, F_2(0), v_2(T_1))\ ,$$

$$v_2(T_1) = \sqrt{\int_0^{T_1} \sigma_2^2(t)dt}\,.$$

where, denoting by Φ the cumulative standard normal distribution function,

$$\begin{aligned}\text{Bl}(K,F,v) &= F\Phi(d_1(K,F,v)) - K\Phi(d_2(K,F,v)),\\ d_1(K,F,v) &= \frac{\ln(F/K) + v^2/2}{v},\\ d_2(K,F,v) &= \frac{\ln(F/K) - v^2/2}{v}.\end{aligned}$$

Clearly, in this derivation, the average volatility of the forward rate in $[0,T_1]$, $v_2(T_1)/\sqrt{T_1}$, does not depend on the strike K of the option. Indeed, volatility is a characteristic of the forward rate underlying the contract, and has nothing to do with the nature of the contract itself, and with the strike K in particular.

Now take two different strikes K_1 and K_2, and suppose that the market provides us with the prices of the two related caplets $\mathbf{Cpl}^{\text{MKT}}(0,T_1,T_2,K_1)$ and $\mathbf{Cpl}^{\text{MKT}}(0,T_1,T_2,K_2)$.[2]

A natural question is now the following. Does there exist a *single* volatility parameter $v_2(T_1)$ such that both

$$\mathbf{Cpl}^{\text{MKT}}(0,T_1,T_2,K_1) = P(0,T_2)\tau\text{Bl}(K_1,F_2(0),v_2(T_1))$$

and

$$\mathbf{Cpl}^{\text{MKT}}(0,T_1,T_2,K_2) = P(0,T_2)\tau\text{Bl}(K_2,F_2(0),v_2(T_1))$$

hold? The answer is a resounding "no" in general. In fact, sticking to Black's formula, two *different* volatilities $v_2(T_1,K_1)$ and $v_2(T_1,K_2)$ are usually required to match the observed market prices:

$$\begin{aligned}\mathbf{Cpl}^{\text{MKT}}(0,T_1,T_2,K_1) &= P(0,T_2)\tau\text{Bl}(K_1,F_2(0),v_2^{\text{MKT}}(T_1,K_1)),\\ \mathbf{Cpl}^{\text{MKT}}(0,T_1,T_2,K_2) &= P(0,T_2)\tau\text{Bl}(K_2,F_2(0),v_2^{\text{MKT}}(T_1,K_2)).\end{aligned}$$

[2] Notice that both caplets have the same underlying forward rates and the same maturity.

The curve $K \mapsto v_2^{\text{MKT}}(T_1, K)/\sqrt{T_1}$ is the so called implied volatility curve of the T_1-expiry caplet.

If Black's formula were consistent along different strikes, this curve would be flat, since volatility should not depend on the strike K. Instead, this curve is commonly seen to be smile- or skew-shaped. Therefore, in order to explain such typical market patterns, one has to resort to alternative dynamics.

Indeed, assume that, under Q^2,

$$dF(t; T_1, T_2) = \nu(t, F(t; T_1, T_2))\, dW_t, \tag{2}$$

where ν can be either a deterministic or a stochastic function of $F(t; T_1, T_2)$. In the latter case we would be using a so called "stochastic-volatility model", where for example $\nu(t, F) = \xi(t)F$, with ξ following a second stochastic differential equation. In this paper, instead, we will concentrate on a deterministic $\nu(t, \cdot)$, thus investigating the class of "local-volatility model".

Our alternative dynamics generates a smile, which is obtained as follows.

1. Set K to a starting value. Compute the model caplet price

   $$\Pi(K) = P(0, T_2)\tau E^2(F(T_1; T_1, T_2) - K)^+$$

   with F obtained through the alternative dynamics (2).

2. Invert Black's formula for this strike, i.e. solve

   $$\Pi(K) = P(0, T_2)\tau \text{Bl}(K, F_2(0), v(K)\sqrt{T_1})$$

   in $v(K)$, thus obtaining the (average) model implied volatility $v(K)$. Then change K and apply this same procedure.

Having alternative dynamics that are not lognormal implies that we obtain a non-flat curve $K \mapsto v(K)$. Clearly, one needs to choose $\nu(t, \cdot)$ flexible enough to be able to resemble the corresponding volatility curves coming from the market.

We finally point out that one has to deal, in general, with an implied-volatility surface, since we have a caplet-volatility curve for each considered expiry. The calibration issues, however, are essentially unchanged, since one can calibrate on each expiry's data separately from the other expiry times.

3 Two Classical Alternative Dynamics

In this section, we first introduce the FLM that can be obtained by displacing a given lognormal diffusion. We then describe the CEV model used by Andersen and Andreasen (2000) to model the evolution of the forward-rate process.

3.1 The Shifted-Lognormal Case

A very simple way of constructing forward-rate dynamics that implies non-flat volatility structures is by shifting the generic lognormal dynamics analogous to (1). Indeed, let us assume that the forward rate

$$F_j := F(\cdot; T_{j-1}, T_j) = \frac{1}{\tau_j}\left[\frac{P(\cdot, T_{j-1})}{P(\cdot, T_j)} - 1\right]$$

evolves, under its associated T_j-forward measure Q^j, according to

$$
\begin{aligned}
F_j(t) &= X_j(t) + \alpha, \\
dX_j(t) &= \beta(t) X_j(t)\, dW_t,
\end{aligned}
\tag{3}
$$

where α is a real constant, β is a deterministic function of time and W is a standard Brownian motion. We immediately have that

$$
dF_j(t) = \beta(t)(F_j(t) - \alpha)\, dW_t, \tag{4}
$$

so that, for $t < T \leq T_{j-1}$, the forward rate F_j can be explicitly written as

$$
F_j(T) = \alpha + (F_j(t) - \alpha) e^{-\frac{1}{2}\int_t^T \beta^2(u)\, du + \int_t^T \beta(u)\, dW_u}. \tag{5}
$$

The distribution of $F_j(T)$, conditional on $F_j(t)$, $t < T \leq T_{j-1}$, is then a shifted lognormal distribution with density

$$
p_{F_j(T)|F_j(t)}(x) = \frac{1}{(x-\alpha)U(t,T)\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left(\frac{\ln\frac{x-\alpha}{F_j(t)-\alpha} + \frac{1}{2}U^2(t,T)}{U(t,T)} \right)^2 \right\}, \tag{6}
$$

for $x > \alpha$, where

$$
U(t,T) := \sqrt{\int_t^T \beta^2(u) du}. \tag{7}
$$

The resulting model for F_j preserves the analytically tractability of the geometric Brownian motion X. Notice indeed that, denoting by E^j the expectation under Q^j,

$$
P(t,T_j) E^j\{[F_j(T_{j-1}) - K]^+ | \mathcal{F}_t\} = P(t,T_j) E^j\{[X_j(T_{j-1}) - (K-\alpha)]^+ | \mathcal{F}_t\},
$$

so that, for $\alpha < K$, the caplet price $\mathbf{Cpl}(t, T_{j-1}, T_j, \tau_j, K)$, with unit notional, associated with (4) is simply given by

$$
\mathbf{Cpl}(t, T_{j-1}, T_j, \tau_j, K) = \tau_j P(t,T_j) \mathrm{Bl}(K-\alpha, F_j(t) - \alpha, U(t, T_{j-1})). \tag{8}
$$

The implied Black volatility $\hat{\sigma} = \hat{\sigma}(K,\alpha)$ corresponding to a given strike K and to a chosen α is obtained by backing out the volatility parameter $\hat{\sigma}$ in Black's formula that matches the model price:

$$
\begin{aligned}
&\tau_j P(t,T_j) \mathrm{Bl}(K, F_j(t), \hat{\sigma}(K,\alpha)\sqrt{T_{j-1} - t}\,) \\
&\quad = \tau_j P(t,T_j) \mathrm{Bl}(K-\alpha, F_j(t) - \alpha, U(t, T_{j-1})).
\end{aligned}
$$

We can now understand why the simple affine transformation (3) can be useful in practice. The resulting forward-rate process, in fact, besides having explicit dynamics and known marginal density, immediately leads to closed-form formulas for caplet prices that allow for skews in the caplet implied volatility. An example of the skewed volatility structure $K \mapsto \hat{\sigma}(K,\alpha)$ that is implied by such a model is shown in Figure 1.[3]

[3] Such a figure shows a decreasing caplet-volatility curve. In real markets, however, different structures can be encountered too (smile-shaped, skewed to the right,...).

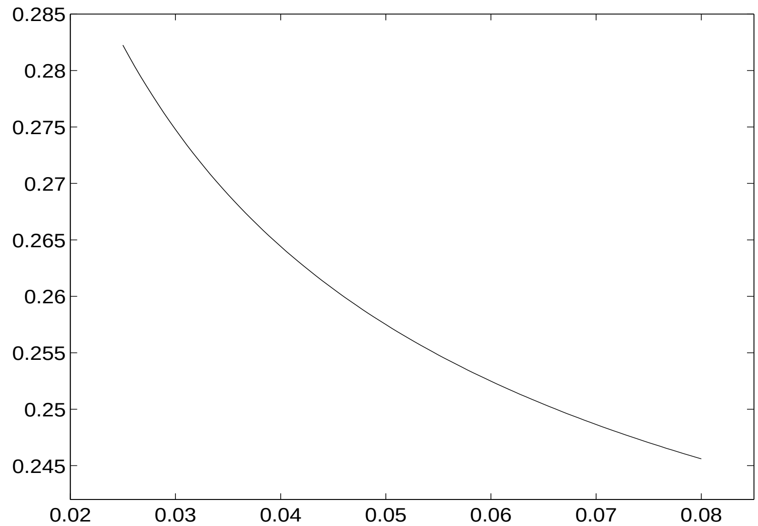

Figure 1: Caplet volatility structure $\hat{\sigma}(K, \alpha)$ plotted against K implied, at time $t = 0$, by the forward-rate dynamics (4), where we set $T_{j-1} = 1$, $T_j = 1.5$, $\alpha = -0.015$, $\beta(t) = 0.2$ for all t and $F_j(0) = 0.055$.

Introducing a non-zero parameter α has two effects on the implied caplet volatility structure, which for $\alpha = 0$ is flat at the constant level $U(0, T_{j-1})$. First, it leads to a strictly decreasing ($\alpha < 0$) or increasing ($\alpha > 0$) curve. Second, it moves the curve upwards ($\alpha < 0$) or downwards ($\alpha > 0$). More generally, ceteris paribus, increasing α shifts the volatility curve $K \mapsto \hat{\sigma}(K, \alpha)$ down, whereas decreasing α shifts the curve up. The formal proof of these properties is straightforward. Notice, for example, that at time $t = 0$ the implied ATM ($K = F_j(0)$) caplet volatility $\hat{\sigma}$ satisfies

$$\mathrm{Bl}(F_j(0), F_j(0), \sqrt{T_{j-1}}\hat{\sigma}(F_j(0), \alpha)) = \mathrm{Bl}(F_j(0) - \alpha, F_j(0) - \alpha, U(0, T_{j-1})),$$

which reads

$$(F_j(0) - \alpha)\left[2\Phi\left(\frac{1}{2}U(0, T_{j-1})\right) - 1\right] = F_j(0)\left[2\Phi\left(\frac{1}{2}\sqrt{T_{j-1}}\hat{\sigma}(F_j(0), \alpha)\right) - 1\right].$$

When increasing α the left hand side of this equation decreases, thus decreasing the $\hat{\sigma}$ in the right-hand side that is needed to match the decreased left-hand side. Moreover, when differentiating (8) with respect to α we obtain a quantity that is always negative.

Shifting a lognormal diffusion can then help in recovering skewed volatility structures. However, such structures are often too rigid, and highly negative slopes are impossible to recover. Moreover, the best fitting of market data is often achieved for decreasing implied volatility curves, which correspond to negative values of the α parameter, and hence to a support of the forward-rate density containing negative values. Even though the probability of negative rates may be negligible in practice, many people regard this drawback as an undesirable feature.

The next models we illustrate may offer the properties and flexibility required for a satisfactory fitting of market data.

3.2 The Constant Elasticity of Variance Model

Another classical model leading to skews in the implied caplet-volatility structure is the CEV model of Cox (1975) and Cox and Ross (1976). Recently, Andersen and Andreasen (2000) applied the CEV dynamics as a model of the evolution of forward LIBOR rates.

Andersen and Andreasen start with a general forward-LIBOR dynamics of the following type, under measure Q^j,

$$dF_j(t) = \phi(F_j(t))\sigma_j(t)\,dW_t,$$

where ϕ is a general function. Andersen and Andreasen suggest as a particularly tractable case in this family the CEV model, where

$$\phi(F_j(t)) = [F_j(t)]^\gamma,$$

with $0 < \gamma < 1$. Notice that the "border" cases $\gamma = 0$ and $\gamma = 1$ would lead respectively to a normal and a lognormal dynamics.

The model then reads

$$dF_j(t) = \sigma_j(t)[F_j(t)]^\gamma\,dW_t, \quad F_j = 0 \ \text{ absorbing boundary when } \ 0 < \gamma < 1/2. \tag{9}$$

For $0 < \gamma < 1/2$ equation (9) does not have a unique solution unless we specify a boundary condition at $F_j = 0$. This is why we take $F_j = 0$ as an absorbing boundary for the above SDE when $0 < \gamma < 1/2$.[4]

Time dependence of σ_j can be dealt with through a deterministic time change. Indeed, by first setting

$$v(\tau, T) = \int_\tau^T \sigma_j(s)^2 ds$$

and then

$$\widetilde{W}(v(0,t)) := \int_0^t \sigma_j(s)dW(s),$$

we obtain a Brownian motion $\widetilde{W}$ with time parameter v. We substitute this time change in equation (9) by setting $f_j(v(t)) := F_j(t)$ and obtain

$$df_j(v) = f_j(v)^\gamma d\widetilde{W}(v), \quad f_j = 0 \ \text{ absorbing boundary when } \ 0 < \gamma < 1/2. \tag{10}$$

This is a process that can be easily transformed into a Bessel process via a change of variable. Straightforward manipulations lead then to the transition density function of f. By also remembering our time change, we can finally go back to the transition density for the continuous part of our original forward-rate dynamics. The continuous part of the density function of $F_j(T)$ conditional on $F_j(t)$, $t < T \le T_{j-1}$, is then given by

$$\begin{aligned} p_{F_j(T)|F_j(t)}(x) &= 2(1-\gamma)k^{1/(2-2\gamma)}(uw^{1-4\gamma})^{1/(4-4\gamma)}e^{-u-w}I_{1/(2-2\gamma)}(2\sqrt{uw}), \\ k &= \frac{1}{2v(t,T)(1-\gamma)^2}, \\ u &= k[F_j(t)]^{2(1-\gamma)}, \\ w &= kx^{2(1-\gamma)}, \end{aligned} \tag{11}$$

[4] Andersen and Andreasen (2000) also extend their treatment to the case $\gamma > 1$, while noticing that this can lead to explosion when leaving the T_j-forward measure (under which the process has null drift).

with I_q denoting the modified Bessel function of the first kind of order q. Moreover, denoting by $g(y,z) = \frac{e^{-z}z^{y-1}}{\Gamma(y)}$ the gamma density function and by $G(y,x) = \int_x^{+\infty} g(y,z)dz$ the complementary gamma distribution, the probability that $F_j(T) = 0$ conditional on $F_j(t)$ is $G\left(\frac{1}{2(1-\gamma)}, u\right)$.

A major advantage of the model (9) is its analytical tractability, allowing for the above transition density function. This transition density can be useful, for example, in Monte Carlo simulations. From knowledge of the density follows also the possibility to price simple claims. In particular, the following explicit formula for a caplet price can be derived:

$$\begin{aligned}\mathbf{Cpl}(t, T_{j-1}, T_j, \tau_j, K) = \tau_j P(t, T_j) \Bigg[F_j(t) \sum_{n=0}^{+\infty} g(n+1, u)\, G\Big(c_n, kK^{2(1-\gamma)}\Big) \\ -K \sum_{n=0}^{+\infty} g\,(c_n, u)\, G\Big(n+1, kK^{2(1-\gamma)}\Big)\Bigg],\end{aligned} \tag{12}$$

where k and u are defined as in (11) and

$$c_n := n + 1 + \frac{1}{2(1-\gamma)}.$$

This price can be expressed also in terms of the non-central chi-squared distribution function we have encountered in the CIR model. Recall that we denote by $\chi^2(x; r, \rho)$ the cumulative distribution function for a non-central chi-squared distribution with r degrees of freedom and non-centrality parameter ρ, computed at point x. Then the above price can be rewritten as

$$\begin{aligned}\mathbf{Cpl}(t, T_{j-1}, T_j, \tau_j, K) = \tau_j P(t, T_j) \Bigg[F_j(t) \left(1 - \chi^2\left(2K^{1-\gamma}; \frac{1}{1-\gamma} + 2, 2u\right)\right) \\ -K\chi^2\left(2u; \frac{1}{1-\gamma}, 2kK^{1-\gamma}\right)\Bigg].\end{aligned} \tag{13}$$

As hinted at above, the caplet price (12) leads to skews in the implied volatility structure. An example of the structure that can be implied is shown in Figure 2. As previously done in the case of a geometric Brownian motion, an extension of the above model can be proposed based on displacing the CEV process (9) and defining accordingly the forward-rate dynamics. The introduction of the extra parameter determining the density shifting may improve the calibration to market data.

Finally, there is the possibly annoying feature of absorption in $F = 0$. While this does not necessarily constitute a problem for caplet pricing, it can be an undesirable feature from an empirical point of view. Also, it is not clear whether there could be some problems when pricing more exotic structures. As a remedy to this absorption problem, Andersen and Andreasen (2000) propose a "Limited" CEV (LCEV) process, where instead of $\phi(F) = F^\gamma$ they set

$$\phi(F) = F \ \min(\epsilon^{\gamma-1}, F^{\gamma-1}),$$

where ϵ is a small positive real number. This function collapses the CEV diffusion coefficient F^γ to a (lognormal) level-proportional diffusion coefficient $F\epsilon^{\gamma-1}$ when F is small enough to make little difference (smaller than ϵ itself). Andersen and Andreasen (2000) compare the LCEV and CEV models as far as cap prices are concerned and conclude that the differences

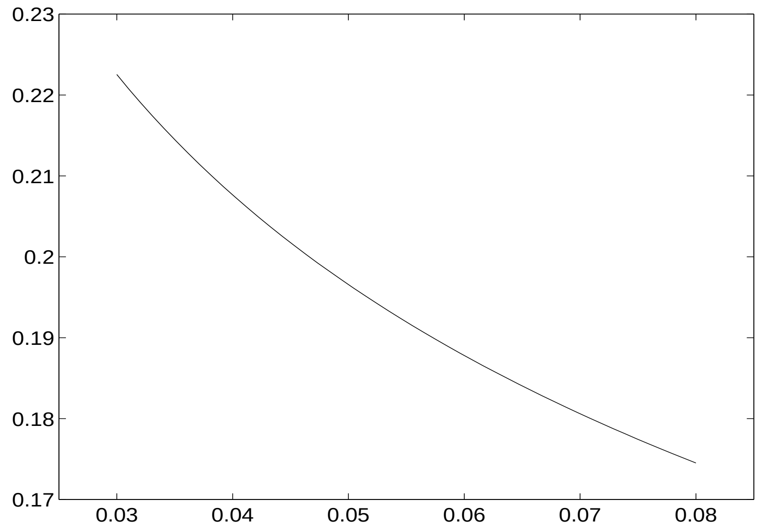

Figure 2: Caplet volatility structure implied by (12) at time $t = 0$, where we set $T_{j-1} = 1$, $T_j = 1.5$, $\sigma_j(t) = 1.5$ for all t, $\gamma = 0.5$ and $F_j(0) = 0.055$.

are small and tend to vanish when $\epsilon \to 0$. They also investigate, to some extent, the speed of convergence. A Crank-Nicholson scheme is used to compute cap prices within the LCEV model. As for the CEV model itself, Andersen and Andreasen allow for $\gamma > 1$ also in the LCEV case, with the difference that then ϵ has to be taken very large.

As far as the calibration of the CEV model to swaptions is concerned, approximated swaption prices based on "freezing the drift" and "collapsing all measures" are also derived. See Andersen and Andreasen (2000) for the details.

4 A Class of Analytically-Tractable Models

We now propose a class of analytically tractable models that are flexible enough to recover a large variety of market volatility structures.

Let the dynamics of the forward rate F_j under the forward measure Q^j be expressed by

$$dF_j(t) = \sigma(t, F_j(t))F_j(t)\, dW_t, \tag{14}$$

where σ is a well-behaved deterministic function.

The function σ, which is usually termed *local volatility* in the financial literature, must be chosen so as to grant a unique strong solution to the SDE (14). In particular, we assume that $\sigma(\cdot,\cdot)$ satisfies, for a suitable positive constant L, the linear-growth condition

$$\sigma^2(t, y)y^2 \le L(1 + y^2) \;\text{ uniformly in } t, \tag{15}$$

which basically ensures existence of a strong solution.

Let us then consider N diffusion processes with dynamics given by

$$dG_i(t) = v_i(t, G_i(t))\, dW_t, \quad i = 1, \dots, N, \quad G_i(0) = F_j(0)\ , \tag{16}$$

with common initial value $F_j(0)$, and where $v_i(t, y)$'s are real functions satisfying regularity conditions to ensure existence and uniqueness of the solution to the SDE (16). In particular we assume that, for suitable positive constants L_i's, the following linear-growth conditions hold:

$$v_i^2(t,y) \le L_i(1+y^2) \text{ uniformly in } t, \quad i=1,\dots,N. \tag{17}$$

For each t, we denote by $p_t^i(\cdot)$ the density function of $G_i(t)$, i.e., $p_t^i(y) = d(Q^j\{G_i(t) \le y\})/dy$, where, in particular, p_0^i is the δ-Dirac function centered in $G_i(0)$.

The problem we want to address is the derivation of the local volatility $\sigma(t, S_t)$ such that the Q^j-density of $F_j(t)$ satisfies, for each time t,

$$p_t(y) := \frac{d}{dy}\, Q^j\{F_j(t) \le y\} = \sum_{i=1}^N \lambda_i \frac{d}{dy}\, Q^j\{G_i(t) \le y\} = \sum_{i=1}^N \lambda_i p_t^i(y), \tag{18}$$

where the λ_i's are strictly positive constants such that $\sum_{i=1}^N \lambda_i = 1$. Indeed, $p_t(\cdot)$ is a proper Q^j-density function since, by definition,

$$\int_0^{+\infty} y p_t(y) dy = \sum_{i=1}^N \lambda_i \int_0^{+\infty} y p_t^i(y) dy = \sum_{i=1}^N \lambda_i G_i(0) = F_j(0).$$

Remark 4.1. *Notice that in the last calculation we were able to recover the proper Q^j-expectation thanks to our assumption that all processes (16) share the same null drift. However, the role of the processes G_i is merely instrumental, and there is no need to assume their drift to be of that form if not for simplifying calculations. In particular, what matters in obtaining the right expectation as in the last formula above is the marginal distribution p_i.*

As already noticed by several authors,[5] the above problem is essentially the reverse to that of finding the marginal density function of the solution of an SDE when the coefficients are known. In particular, $\sigma(t, F_j(t))$ can be found by solving the Fokker-Planck equation

$$\frac{\partial}{\partial t} p_t(y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(t,y) y^2 p_t(y)\right), \tag{19}$$

given that each density $p_t^i(y)$ satisfies itself the Fokker-Planck equation

$$\frac{\partial}{\partial t} p_t^i(y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(v_i^2(t,y) p_t^i(y)\right). \tag{20}$$

Applying the definition (18) and the linearity of the derivative operator, (19) can be written as

$$\sum_{i=1}^N \lambda_i \frac{\partial}{\partial t} p_t^i(y) = \sum_{i=1}^N \lambda_i \left[-\frac{\partial}{\partial y}\left(\mu y p_t^i(y)\right)\right] + \sum_{i=1}^N \lambda_i \left[\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(t,y) y^2 p_t^i(y)\right)\right],$$

which by substituting from (20) becomes

$$\sum_{i=1}^N \lambda_i \left[\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(v_i^2(t,y) p_t^i(y)\right)\right] = \sum_{i=1}^N \lambda_i \left[\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(t,y) y^2 p_t^i(y)\right)\right].$$

[5] See for instance Dupire (1997).

Using again linearity of the second order derivative operator, we obtain

$$\frac{\partial^2}{\partial y^2}\left[\sum_{i=1}^{N}\lambda_i v_i^2(t,y)p_t^i(y)\right] = \frac{\partial^2}{\partial y^2}\left[\sigma^2(t,y)y^2\sum_{i=1}^{N}\lambda_i p_t^i(y)\right].$$

If we look at this last equation as to a second order differential equation for $\sigma(t,\cdot)$, we find easily its general solution

$$\sigma^2(t,y)y^2\sum_{i=1}^{N}\lambda_i p_t^i(y) = \sum_{i=1}^{N}\lambda_i v_i^2(t,y)p_t^i(y) + A_t y + B_t, \tag{21}$$

with A and B suitable real functions of time. The regularity conditions (17) and (15) imply that the LHS of the equation has zero limit for $y \to \infty$. As a consequence, the RHS must have a zero limit as well. This holds if and only if $A_t = B_t = 0$, for each t. We therefore obtain that the expression for $\sigma(t,y)$ that is consistent with the marginal density (18) and with the regularity constraint (15) is, for $(t,y) > (0,0)$,

$$\sigma(t,y) = \sqrt{\frac{\sum_{i=1}^{N}\lambda_i v_i^2(t,y)p_t^i(y)}{\sum_{i=1}^{N}\lambda_i y^2 p_t^i(y)}}. \tag{22}$$

Indeed, notice that by setting

$$\Lambda_i(t,y) := \frac{\lambda_i p_t^i(y)}{\sum_{i=1}^{N}\lambda_i p_t^i(y)} \tag{23}$$

for each $i = 1,\dots,N$ and $(t,y) > (0,0)$, we can write

$$\sigma^2(t,y) = \sum_{i=1}^{N}\Lambda_i(t,y)\frac{v_i^2(t,y)}{y^2}, \tag{24}$$

so that the square of the volatility σ can be written as a (stochastic) convex combination of the squared volatilities of the basic processes (16). In fact, for each (t,y), $\Lambda_i(t,y) \geq 0$ for each i and $\sum_{i=1}^{N}\Lambda_i(t,y) = 1$. Moreover, by (17) and setting $L := \max_{i=1,\dots,N} L_i$, the condition (15) is fulfilled since

$$\sigma^2(t,y)y^2 = \sum_{i=1}^{N}\Lambda_i(t,y)v_i^2(t,y) \leq \sum_{i=1}^{N}\Lambda_i(t,y)L_i(1+y^2) \leq L(1+y^2).$$

The function σ may be then extended to the semi-axes $\{(t,0) : t > 0\}$ and $\{(0,y) : y > 0\}$ according to the specific choice of the basic densities $p_t^i(\cdot)$.

Formula (22) leads to the following SDE for the forward rate under measure Q^j:

$$dF_j(t) = \sqrt{\frac{\sum_{i=1}^{N}\lambda_i v_i^2(t,F_j(t))p_t^i(F_j(t))}{\sum_{i=1}^{N}\lambda_i F_j(t)^2 p_t^i(F_j(t))}}F_j(t)\,dW_t. \tag{25}$$

This SDE, however, must be regarded as defining some candidate dynamics that leads to the marginal density (18). Indeed, if σ is bounded, then the SDE is well defined, but the conditions we have imposed so far are not sufficient to grant the uniqueness of the strong solution, so that a verification must be done on a case-by-case basis.

Let us now assume that the SDE (25) has a unique strong solution. We will see later on a fundamental case where this assumption holds. Then, remembering the definition (18), it is straightforward to derive the model caplet prices in terms of the caplet prices associated to the basic models (16). Indeed, let us consider a caplet with strike K associated to the given forward rate. Then, the caplet price at the initial time $t = 0$ is given by

$$\begin{aligned}\mathbf{Cpl}(0, T_{j-1}, T_j, \tau_j, K) &= \tau_j P(0, T_j) E^j \left\{ [F_j(T) - K]^+ \right\} \\ &= \tau_j P(0, T_j) \sum_{i=1}^{N} \lambda_i \int_0^{+\infty} [y - K]^+ p^i_{T_j}(y) dy \\ &= \sum_{i=1}^{N} \lambda_i \mathbf{Cpl}^i(0, T_{j-1}, T_j, \tau_j, K),\end{aligned} \tag{26}$$

where $\mathbf{Cpl}^i(0, T_{j-1}, T_j, \tau_j, K)$ denotes the caplet price, with unit notional amount, associated with (16).

We can now justify our assumption that the forward rate marginal density be given by the mixture of known basic densities. When proposing alternative dynamics, it is usually quite problematic to come up with analytical formulas for caplets. Here, instead, such problem can be avoided since the beginning if we use analytically-tractable densities p^i.[6] Moreover, the absence of bounds on the parameter N implies that a virtually unlimited number of parameters can be introduced in the dynamics so as to be used for a better calibration to market data.

A last remark concerns the classical economic interpretation of a mixture of densities. We can indeed view F_j as a process whose density at time t coincides with the basic density p^i_t with probability λ_i.

5 A Mixture-of-Lognormals Model

Let us now consider the particular case where the densities p^i_t's are all lognormal. Precisely, we assume that, for each i,

$$v_i(t, y) = \sigma_i(t) y, \tag{27}$$

where all σ_i's are deterministic and continuous functions of time that are bounded from above and below by (strictly) positive constants.

The marginal density of $G_i(t)$, for each time t, is then lognormal and given by

$$\begin{aligned} p^i_t(y) &= \frac{1}{y V_i(t) \sqrt{2\pi}} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln \frac{y}{F_j(0)} + \tfrac{1}{2} V_i^2(t) \right]^2 \right\}, \\ V_i(t) &:= \sqrt{\int_0^t \sigma_i^2(u) du}. \end{aligned} \tag{28}$$

Brigo and Mercurio (2001a) proved the following.

[6] Note that, due to the linearity of the derivative operator, the same convex combination applies to all Greeks.

Proposition 5.1. *Let us assume there exists an $\varepsilon > 0$ such that $\sigma_i(t) = \sigma_0 > 0$, for each t in $[0, \varepsilon]$ and $i = 1, \dots, N$. Then, if we set*

$$\nu(t,y) := \sqrt{\frac{\sum_{i=1}^N \lambda_i \sigma_i^2(t) \frac{1}{V_i(t)} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{F_j(0)} + \frac{1}{2}V_i^2(t)\right]^2\right\}}{\sum_{i=1}^N \lambda_i \frac{1}{V_i(t)} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{F_j(0)} + \frac{1}{2}V_i^2(t)\right]^2\right\}}}, \tag{29}$$

for $(t, y) > (0, 0)$ and $\nu(t, y) = \sigma_0$ for $(t, y) = (0, F_j(0))$, the SDE

$$dF_j(t) = \nu(t, F_j(t)) F_j(t)\, dW_t \tag{30}$$

has a unique strong solution whose marginal density is given by the mixture of lognormals

$$p_t(y) = \sum_{i=1}^N \lambda_i \frac{1}{y V_i(t)\sqrt{2\pi}} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{F_j(0)} + \frac{1}{2}V_i^2(t)\right]^2\right\} \tag{31}$$

The above proposition provides us with the analytical expression for the diffusion coefficient in the SDE (14) such that the resulting equation has a unique strong solution whose marginal density is given by (31).

The square of the local volatility $\nu(t, y)$ can be viewed as a weighted average of the squared basic volatilities $\sigma_1^2(t), \dots, \sigma_N^2(t)$, where the weights are all functions of the lognormal marginal densities (28). That is, for each $(t, y) > (0, 0)$, we can write

$$\nu^2(t,y) = \sum_{i=1}^N \Lambda_i(t,y)\sigma_i^2(t),$$
$$\Lambda_i(t,y) := \frac{\lambda_i p_t^i(y)}{\sum_{i=1}^N \lambda_i p_t^i(y)}.$$

As a consequence, for each $t > 0$ and $y > 0$, the function ν is bounded from below and above by (strictly) positive constants. In fact

$$\sigma_* \le \nu(t,y) \le \sigma^* \quad \text{for each } t, y > 0, \tag{32}$$

where

$$\sigma_* := \inf_{t \ge 0}\left\{\min_{i=1,\dots,N} \sigma_i(t)\right\} > 0,$$
$$\sigma^* := \sup_{t \ge 0}\left\{\max_{i=1,\dots,N} \sigma_i(t)\right\} < +\infty.$$

Remark 5.2. *The function $\nu(t, y)$ can be extended by continuity to the semi-axes $\{(0, y) : y > 0\}$ and $\{(t, 0) : t \ge 0\}$ by setting $\nu(0, y) = \sigma_0$ and $\nu(t, 0) = \nu^*(t)$, where $\nu^*(t) := \sigma_{i^*}(t)$ and $i^* = i^*(t)$ is such that $V_{i^*}(t) = \max_{i=1,\dots,N} V_i(t)$. In particular, $\nu(0, 0) = \sigma_0$. Indeed, for every $\bar{y} > 0$ and every $\bar{t} \ge 0$,*

$$\lim_{t \to 0} \nu(t, \bar{y}) = \sigma_0,$$
$$\lim_{y \to 0} \nu(\bar{t}, y) = \nu^*(t).$$

At time $t = 0$, the pricing of caplets under our forward-rate dynamics (30) is quite straightforward. Indeed,

$$\begin{aligned} P(0,T_j)E^j\{[F_j(T_{j-1})-K]^+\} &= P(0,T_j)\int_0^{+\infty}(y-K)^+ p_{T_{j-1}}(y)dy \\ &= P(0,T_j)\sum_{i=1}^{N}\lambda_i\int_0^{+\infty}(y-K)^+ p^i_{T_{j-1}}(y)dy \end{aligned}$$

so that, the caplet price $\mathbf{Cpl}(0,T_{j-1},T_j,\tau_j,K)$ associated with our dynamics (30) is simply given by

$$\mathbf{Cpl}(0,T_{j-1},T_j,\tau_j,N,K) = \tau_j P(0,T_j)\sum_{i=1}^{N}\lambda_i \mathrm{Bl}(K,F_j(0),V_i(T_{j-1})). \tag{33}$$

The caplet price (33) leads to smiles in the implied volatility structure. An example of the shape that can be reproduced is shown in Figure 3.[7] Observe that the implied volatility curve has a minimum exactly at a strike equal to the initial forward rate $F_j(0)$. This property, which is formally proven in Brigo and Mercurio (2001a), makes the model suitable for recovering smile-shaped volatility surfaces. In fact, also skewed shapes can be retrieved, but with zero slope at the ATM level.

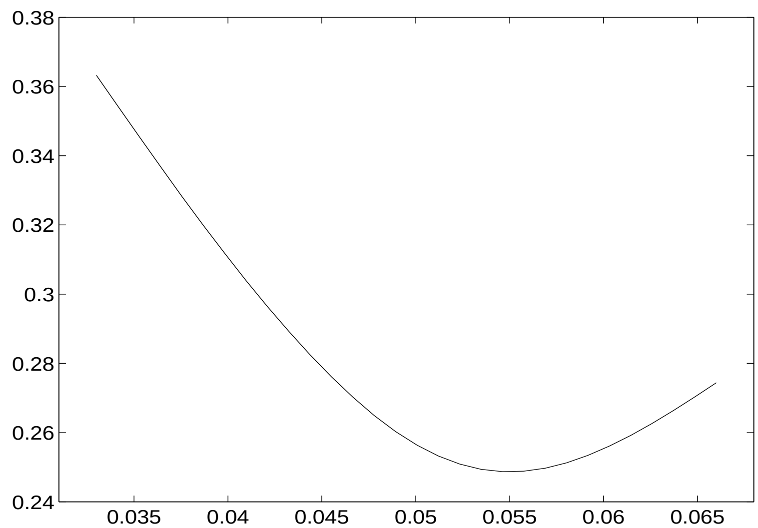

Figure 3: Caplet volatility structure implied by the option prices (33), where we set $T_{j-1} = 1$, $N = 3$, $(V_1(1), V_2(1), V_3(1)) = (0.6, 0.1, 0.2)$, $(\lambda_1, \lambda_2, \lambda_3) = (0.2, 0.3, 0.5)$ and $F_j(0) = 0.055$.

Given the above analytical tractability, we can easily derive an explicit approximation for the caplet implied volatility as a function of the caplet strike price. More precisely, define the moneyness m as the logarithm of the ratio between the forward rate and the strike, i.e.,

$$m := \ln\frac{F_j(0)}{K}\,.$$

[7]In such a figure, we consider directly the values of the V_i's. Notice that one can easily find some σ_i's satisfying our technical assumptions that are consistent with the chosen V_i's.

The implied volatility $\hat{\sigma}(m)$ for the moneyness m is implicitly defined by equating the Black caplet price in $\hat{\sigma}(m)$ to the price implied by our model according to

$$\begin{aligned}&\left[\Phi\left(\frac{m+\frac{1}{2}\hat{\sigma}(m)^2T_{j-1}}{\hat{\sigma}(m)\sqrt{T_{j-1}}}\right)-e^{-m}\Phi\left(\frac{m-\frac{1}{2}\hat{\sigma}(m)^2T_{j-1}}{\hat{\sigma}(m)\sqrt{T_{j-1}}}\right)\right]\\&\quad=\sum_{i=1}^{N}\lambda_i\left[\Phi\left(\frac{m+\frac{1}{2}V_i^2(T_{j-1})}{V_i(T_{j-1})}\right)-e^{-m}\Phi\left(\frac{m-\frac{1}{2}V_i^2(T_{j-1})}{V_i(T_{j-1})}\right)\right].\end{aligned} \tag{34}$$

A repeated application of Dini's implicit function theorem and a Taylor's expansion around $m = 0$, lead to

$$\hat{\sigma}(m)=\hat{\sigma}(0)+\frac{1}{2\hat{\sigma}(0)T_{j-1}}\sum_{i=1}^{N}\lambda_i\left[\frac{\hat{\sigma}(0)\sqrt{T_{j-1}}}{V_i(T_{j-1})}e^{\frac{1}{8}\left(\hat{\sigma}(0)^2T_{j-1}-V_i^2(T_{j-1})\right)}-1\right]m^2+o(m^3) \tag{35}$$

where the ATM implied volatility, $\hat{\sigma}(0)$, is explicitly given by

$$\hat{\sigma}(0)=\frac{2}{\sqrt{T_{j-1}}}\,\Phi^{-1}\left(\sum_{i=1}^{N}\lambda_i\Phi\left(\frac{1}{2}V_i(T_{j-1})\right)\right). \tag{36}$$

5.1 Forward Rates Dynamics under Different Measures

So far we have just considered the dynamics of a single forward rate F_j under its canonical measure Q^j. Indeed, as far as calibration issues are concerned this is all that matters.

However, in order to price exotic derivatives, one typically needs to propagate the whole term structure of rates under a common reference measure.[8] This is why we need the following.

Let $t = 0$ be the current time. Consider a set $\{T_0, \ldots, T_M\}$ from which expiry-maturity pairs of dates (T_{i-1}, T_i) for a family of spanning forward rates are taken. We shall denote by $\{\tau_1, \ldots, \tau_M\}$ the corresponding year fractions, meaning that τ_i is the year fraction associated with the expiry-maturity pair (T_{i-1}, T_i) for $i > 0$. Times T_i will be usually expressed in years from the current time.

Proposition 5.1 applies to any forward rate, provided one consider different coefficients for different rates. Precisely, assume $\sigma_{i,j}$'s are deterministic and continuous functions of time that are bounded from above and below by (strictly) positive constants, and that there exists an $\varepsilon > 0$ such that $\sigma_{i,j}(t) = \sigma_j^0 > 0$, for each t in $[0, \varepsilon]$ and $i = 1, \ldots, N$. Then define

$$\begin{aligned}V_{i,j}(t) &:= \sqrt{\int_0^t \sigma_{i,j}^2(u)du}\\ \nu_j(t,y) &:= \sqrt{\frac{\sum_{i=1}^{N}\lambda_{i,j}\,\sigma_{i,j}^2(t)\frac{1}{V_{i,j}(t)}\exp\left\{-\frac{1}{2V_{i,j}^2(t)}\left[\ln\frac{y}{F_j(0)}+\frac{1}{2}V_{i,j}^2(t)\right]^2\right\}}{\sum_{i=1}^{N}\lambda_{i,j}\frac{1}{V_{i,j}(t)}\exp\left\{-\frac{1}{2V_{i,j}^2(t)}\left[\ln\frac{y}{F_j(0)}+\frac{1}{2}V_{i,j}^2(t)\right]^2\right\}}},\end{aligned}$$

with $\lambda_{i,j} > 0$, for each i, j, and $\sum_{i=1}^{N}\lambda_{i,j} = 1$ for each j.

[8]The forward measure associated to the derivative final maturity is often the most convenient choice.

Proposition 5.3. *The dynamics of $F_j = F(\cdot; T_{j-1}, T_j)$ under the forward measure Q^i in the three cases $i < j$, $i = j$ and $i > j$ are, respectively,*

$i < j,\ t \le T_i$:

$$dF_j(t) = \nu_j(t, F_j(t))F_j(t) \sum_{k=i+1}^{j} \frac{\rho_{j,k}\,\tau_k\,\nu_k(t, F_k(t))\,F_k(t)}{1 + \tau_k F_k(t)}\,dt + \nu_j(t, F_j(t))F_j(t)\,dW_j^i(t),$$

$i = j,\ t \le T_{j-1}$:

$$dF_j(t) = \nu_j(t, F_j(t))F_j(t)\,dW_j^i(t),$$

$i > j,\ t \le T_{j-1}$:

$$dF_j(t) = -\nu_j(t, F_j(t))F_j(t) \sum_{k=j+1}^{i} \frac{\rho_{j,k}\,\tau_k\,\nu_k(t, F_k(t))\,F_k(t)}{1 + \tau_k F_k(t)}\,dt + \nu_j(t, F_j(t))F_j(t)\,dW_j^i(t),$$

where $W^i = (W_1^i, \ldots, W_M^i)$ is an M-dimensional Brownian motion under Q^i, with instantaneous correlation matrix $(\rho_{j,k})$, meaning that $dW_j^i(t)\,dW_k^i(t) = \rho_{j,k}\,dt$.

Moreover, all of the above equations admit a unique strong solution.

Proof. The proof is a direct consequence of Proposition 6.3.1. in Brigo and Mercurio (2001b) and of the fact that all volatility coefficients ν_j's are bounded. □

Remark 5.4 (Swaptions pricing). *In order to analytically price swaptions the classical "freezing-the-drift" technique can be employed for deriving analytical approximations of implied swaption volatilities.*[9] *For instance, in the case $i < j$, the above forward rate dynamics can be substituted by*

$$dF_j(t) = \nu_j(t, F_j(0))F_j(t) \sum_{k=i+1}^{j} \frac{\rho_{j,k}\,\tau_k\,\nu_k(t, F_k(0))\,F_k(0)}{1 + \tau_k F_k(0)}\,dt + \nu_j(t, F_j(0))F_j(t)\,dW_j^i(t).$$

6 Shifting the Lognormal-Mixture Dynamics

Brigo and Mercurio (2000, 2001b) proposed a simple way to generalize the dynamics (30). With the main target consisting of retrieving a larger variety of volatility structures, the basic lognormal-mixture model was combined with the displaced-diffusion technique by assuming that the forward-rate process F_j is given by

$$F_j(t) = \alpha + \bar{F}_j(t), \tag{37}$$

where α is a real constant and $\bar{F}_j$ evolves according to the basic "lognormal mixture" dynamics (30). It is easy to prove that this is actually the most general affine transformation for which the forward-rate process is still a martingale under its canonical measure.

Dropping the index j where is redundant, so as to come back to the initial notation of Section 5, the analytical expression for the marginal density of such a process is given by the

[9] We refer to Brigo and Mercurio (2001b) for an exhaustive explanation and justification of this methodology.

shifted mixture of lognormals

$$p_t(y) = \sum_{i=1}^{N} \lambda_i \frac{1}{(y-\alpha)V_i(t)\sqrt{2\pi}} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln \frac{y-\alpha}{F_j(0)-\alpha} + \tfrac{1}{2}V_i^2(t) \right]^2 \right\},$$

with $y > \alpha$.

By Ito's formula, we obtain that the forward rate process evolves according to

$$dF_j(t) = \nu(t, F_j(t)-\alpha)\,(F_j(t)-\alpha)\,dW_t. \tag{38}$$

This model for the forward rate process preserves the analytical tractability of the original process $\bar{F}_j$. Indeed,

$$P(0,T_j)E^j\left\{[F_j(T_{j-1})-K]^+\right\} = P(0,T_j)E^j\left\{\left[\bar{F}(T_{j-1})-(K-\alpha)\right]^+\right\},$$

so that, for $\alpha < K$, the caplet price $\mathbf{Cpl}(0, T_{j-1}, T_j, \tau_j, K)$ associated with (37) is simply given by

$$\mathbf{Cpl}(t, T_{j-1}, T_j, \tau_j, K) = \tau_j P(t, T_j) \sum_{i=1}^{N} \lambda_i \mathrm{Bl}(K-\alpha, F_j(0)-\alpha, V_i(T_{j-1})). \tag{39}$$

Moreover, the caplet implied volatility (as a function of m) can be approximated as follows:

$$\begin{aligned}
\hat{\sigma}(m) &= \hat{\sigma}(0) + \hat{\sigma}'(0)m + \frac{1}{2}\hat{\sigma}''(0)m^2 + o(m^2) \\
\hat{\sigma}'(0) &= \alpha\sqrt{\frac{2\pi}{T_{j-1}}}\frac{e^{\frac{1}{8}\hat{\sigma}(0)^2 T_{j-1}}}{F_j(0)}\left(-\sum_{i=1}^{N}\lambda_i\Phi\left(\frac{1}{2}V_i(T_{j-1})\right) + \frac{1}{2}\right) \\
\hat{\sigma}''(0) &= \frac{F_j(0)}{F_j(0)-\alpha}\sum_{i=1}^{N}\lambda_i\frac{e^{\frac{1}{8}(\hat{\sigma}(0)^2 T_{j-1} - V_i(T_{j-1})^2)}}{V_i(T_{j-1})\sqrt{T_{j-1}}} - \frac{4-\hat{\sigma}(0)^2\hat{\sigma}'(0)^2 T_{j-1}^2}{4\hat{\sigma}(0)T_{j-1}},
\end{aligned}$$

where the ATM implied volatility, $\hat{\sigma}(0)$, is explicitly given by

$$\hat{\sigma}(0) = \frac{2}{F_j(0)\sqrt{T_{j-1}}}\Phi^{-1}\left((F_j(0)-\alpha)\sum_{i=1}^{N}\lambda_i\Phi\left(\frac{1}{2}V_i(T_{j-1})\right) + \frac{\alpha}{2}\right).$$

For $\alpha = 0$ the process F_j obviously coincides with $\bar{F}_j$ while preserving the correct zero drift. The introduction of the new parameter α has the effect that, decreasing α, the variance of the asset price at each time increases while maintaining the correct expectation. Indeed:

$$\begin{aligned}
E(F_j(t)) &= F_j(0), \\
\mathrm{Var}(F_j(t)) &= (F_j(0)-\alpha)^2\left(\sum_{i=1}^{N}\lambda_i e^{V_i^2(t)} - 1\right).
\end{aligned}$$

As for the model (3), the parameter α affects the shape of the implied volatility curve in two ways. First, it concurs to determine the level of such curve in that changing α leads to an almost parallel shift of the curve itself. Second, it moves the strike with minimum volatility.

Precisely, if $\alpha > 0$ (< 0) the minimum is attained for strikes lower (higher) than the ATM's. When varying all parameters, the parameter α can be used to add asymmetry around the ATM volatility without shifting the curve.

Finally, as far as the calibration of the above models to swaptions is concerned, once again approximated swaption prices similar to Rebonato's formula in the FLM and based on "freezing the drift" and "collapsing all measures" approaches can be attempted, although results need to be checked numerically in a sufficiently rich number of situations.

7 Two Further Alternative Dynamics

We now consider two further examples in the class of Section 4. The resulting processes, though slightly more involved than (30), have the major advantage of being more flexible as far as the implied caplet volatility curves are concerned.

7.1 A Lognormal-Mixture with Different Means

In the first example we consider, the densities p_t^i's are still lognormal, but their means are now assumed to be different. Precisely, we assume that the instrumental processes G_i evolve, under Q^j, according to

$$dG_i(t) = \mu_i(t)G_i(t)dt + \sigma_i(t)G_i(t)\,dW_t, \quad i = 1, \dots, N, \quad G_i(0) = F_j(0)\ ,$$

where σ_i's satisfy the conditions of Section 5, and μ_i's are deterministic functions of time. The density of G_i at time t is thus given by

$$\begin{aligned} p_t^i(y) &= \frac{1}{yV_i(t)\sqrt{2\pi}} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{F_j(0)} - M_i(t) + \tfrac{1}{2}V_i^2(t)\right]^2\right\}, \\ M_i(t) &:= \int_0^t \mu_i(u)du, \end{aligned} \tag{40}$$

with V_i defined as before. The functions μ_i's can not be defined arbitrarily, but must be chosen so that

$$\sum_{i=1}^N \lambda_i e^{M_i(t)} = 1, \quad \forall t > 0. \tag{41}$$

This is because $p_t(y) = \sum_{i=1}^N \lambda_i p_t^i(y)$ must have a constant mean equal to $F_j(0)$.

As in Section 4, we look for a diffusion coefficient $\psi(\cdot,\cdot)$ such that

$$dF_j(t) = \psi(t, F_j(t))F_j(t)\,dW_t \tag{42}$$

has a solution with marginal density $p_t(y) = \sum_{i=1}^N \lambda_i p_t^i(y)$. As before, we then use the Fokker-Planck equations for processes F_j and G_i's to find that

$$\begin{aligned} \psi(t,y)^2 &:= \nu(t,y)^2 + \frac{2\sum_{i=1}^N \lambda_i\mu_i(t)\int_y^{+\infty} x p_t^i(x)dx}{y^2\sum_{i=1}^N \lambda_i p_t^i(y)} \\ &= \nu(t,y)^2 + \frac{2F_j(0)\sum_{i=1}^N \lambda_i\mu_i(t)\,e^{M_i(t)}\Phi\!\left(\frac{\ln\frac{F_j(0)}{y} + M_i(t) + \frac{1}{2}V_i^2(t)}{V_i(t)}\right)}{y^2\sum_{i=1}^N \lambda_i p_t^i(y)}\ , \end{aligned} \tag{43}$$

with ν defined as in (29), namely

$$\nu(t,y)^2 = \frac{\sum_{i=1}^{N} \lambda_i \sigma_i(t)^2 p_t^i(y)}{\sum_{i=1}^{M} \lambda_i p_t^i(y)},$$

where the new p_t^i's are to be used.[10] The coefficient ψ is not necessarily well defined, since the second term in the RHS of (43) can become negative for some choices of the basic parameters. However, Brigo et al. (2002) have derived conditions under which the strict positivity of $\psi(t,y)^2$ is granted. Such conditions are reported in the following.

Lemma 7.1. *Assume that:*

i) there exists $n \in \{1,2,\ldots,N\}$ such that, for each $t \in [0,T_{j-1}]$, $\mu_i(t) \geq 0$ for each $i = 1,\ldots,N$, $i \neq n$, and $\mu_n(t) \leq 0$;

ii) the condition

$$\frac{V_i^2(t)}{2} - \frac{2V_i^2(t)}{\sigma_i^2(t)}\mu_i(t) > \frac{V_n^2(t)}{2} - \frac{2V_n^2(t)}{\sigma_n^2(t)}\mu_n(t) \tag{44}$$

is satisfied for each $t \in (0,T_{j-1}]$ and for each $i \neq n$.

Then the function ψ^2 in (43) is strictly positive on $(0,T_{j-1}] \times (0,+\infty)$.

Brigo et al. (2002) also proved the following result concerning the existence and uniqueness of the solution to the SDE (42).

Proposition 7.2. *Let us assume that each σ_i is continuous and bounded from below by a positive constant, and that there exists an $\varepsilon > 0$ such that $\sigma_i(t) = \sigma_0 > 0$, for each t in $[0,\varepsilon]$ and $i = 1,\ldots,N$. Let us further assume that each μ_i is continuous, that (41) is satisfied, and that $\mu_i(t) = \mu > 0$, for each t in $[0,\varepsilon]$ and $i = 1,\ldots,N$. Then, under the assumptions of Lemma 7.1, the SDE (42) has a unique strong solution whose marginal density is given by the mixture of lognormal densities (40).*

The pricing of caplets, under dynamics (42), is again quite straightforward. Indeed, the caplet price $\mathbf{Cpl}(0,T_{j-1},T_j,\tau_j,K)$ is simply given by

$$\mathbf{Cpl}(0,T_{j-1},T_j,\tau_j,K) = \tau_j P(0,T_j) \sum_{i=1}^{N} \lambda_i e^{M_i(T_{j-1})} \mathrm{Bl}\left(K\, e^{-M_i(T_{j-1})}, F_j(0), V_i(T_{j-1})\right). \tag{45}$$

Also this price leads to smiles in the implied volatility structure. However, the non-zero drifts in the G_i-dynamics allows us to reproduce steeper and more skewed curves than in the zero-drifts case, with minimums that can be shifted far away from the ATM level.

7.2 The Case of Hyperbolic-Sine Processes

The second case we consider lies in the class of dynamics (25). We in fact assume that the basic processes G_i evolve, under Q^j, according to a hyperbolic-sine process, i.e.[11]

$$G_i(t) = \beta_i(t) \sinh\left[\int_0^t \alpha_i(u)dW_u - L_i\right], \quad i = 1,\ldots,N, \quad G_i(0) = F_j(0), \tag{46}$$

[10]We notice that the integrals in the numerator of the second term in the RHS of (43) are quantities proportional the Black-Scholes prices of asset or nothing options for the instrumental processes G_i.

[11]We remind that $\sinh(x) = \frac{e^x - e^{-x}}{2}$, and that $\sinh^{-1}(x) = \ln(x + \sqrt{1+x^2})$.

where α_i's are positive and deterministic functions of time, L_i's are negative constants, and β_i's are chosen so as to render the G_i's martingales, namely

$$\beta_i(t) = \frac{F_j(0)\, e^{-\frac{1}{2}A_i^2(t)}}{\sinh(-\alpha_i L_i)},$$

where we set $A_i(t) := \sqrt{\int_0^t \alpha_i^2(u)du}$.

Each S_i is thus defined as an increasing function of a time-changed Brownian motion, and evolves according to

$$dG_i(t) = \alpha_i(t)\sqrt{\beta_i^2(t) + G_i^2(t)}\, dW_t, \quad i = 1, \dots, N.$$

Looking at this SDE's diffusion coefficient we immediately notice that it is roughly deterministic for small values of $G_i(t)$, whereas it is roughly proportional to $G_i(t)$ for large values of $G_i(t)$. Therefore in the former case, the dynamics are approximately of Gaussian type, whereas in the latter they are approximately of lognormal type. For further details on such a process we refer to Carr et al. (1999).[12]

The hyperbolic-sine process (46) shares all the analytical tractability of the classical geometric Brownian motion. This is intuitive, since (46) is basically the difference of two geometric Brownian motions (with perfectly negatively correlated logarithms).

The cumulative distribution function of process G_i at each time t is easily derived as follows:

$$\begin{aligned} Q^j\{G_i(t) \le y\} &= Q^j\left\{\int_0^t \alpha_i(u)dW_u \le L_i + \sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right\} \\ &= \Phi\left(\frac{L_i}{A_i(t)} + \frac{1}{A_i(t)}\sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right), \end{aligned}$$

so that the time-t marginal density of G_i is

$$p_t^i(y) = \frac{\exp\left\{-\frac{1}{2A_i^2(t)}\left[L_i + \sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right]^2\right\}}{A_i(t)\sqrt{2\pi}\sqrt{\beta_i^2(t) + y^2}}. \tag{47}$$

Moreover, through a straightforward integration, we obtain the associated caplet price as

$$\begin{aligned} \mathbf{Cpl}(0, T_{j-1}, T_j, \tau_j, K) = \tau_j P(0, T_j)\Bigg[&\frac{F_j(0)}{2\sinh(-L_i)}\Big(e^{-L_i}\Phi\left(\bar{y}_i(T_{j-1}) + A_i(T_{j-1})\right) \\ &- e^{L_i}\Phi\left(\bar{y}_i(T_{j-1}) - A_i(T_{j-1})\right)\Big) - K\Phi\big(\bar{y}_i(T_{j-1})\big)\Bigg], \end{aligned} \tag{48}$$

where we set, for a general T,

$$\bar{y}_i(T) := -\frac{L_i}{A_i(T)} - \frac{1}{A_i(T)}\sinh^{-1}\left(\frac{K}{\beta_i(T)}\right).$$

[12]Carr et al. (1999) actually consider a process where negative values are absorbed into zero. Their process is slightly more complicated, but not lose in analytical tractability.

The pricing function (48) leads to steeply decreasing patterns in the implied volatility curve. Therefore, we can hope that a mixture of densities (47) leads to steeper implied volatility skews than in the lognormal-mixture model. Indeed, this turns out to be the case.

The results in Section 4, and equation (25) in particular, immediately yield the following SDE for the forward rate under measure Q^j:

$$dF_j(t) = \chi(t, F_j(t))\, dW_t$$

$$\chi(t,y) := \sqrt{\frac{\sum_{i=1}^N \lambda_i \frac{\alpha_i^2(t)\sqrt{\beta_i(t)^2+y^2}}{A_i(t)} \exp\left\{-\frac{1}{2A_i^2(t)}\left[L_i + \sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right]^2\right\}}{\sum_{i=1}^N \frac{\lambda_i}{A_i(t)\sqrt{\beta_i(t)^2+y^2}} \exp\left\{-\frac{1}{2A_i^2(t)}\left[L_i + \sinh^{-1}\left(\frac{y}{\beta_i(t)}\right)\right]^2\right\}}} \qquad (t,y) > (0,0). \tag{49}$$

As in the previous lognormal-mixtures cases, this equation must be handled with due care since the function χ is in general discontinuous in $(0, F_j(0))$. However, Brigo et al. (2002) has proven the following result stating the existence and uniqueness of a solution of such SDE under mild assumptions on the model coefficients.

Proposition 7.3. *Let us assume that each α_i is continuous and bounded from below by a positive constant, that there exists an $\varepsilon > 0$ such that $\alpha_i(t) = \alpha_0 > 0$, for each t in $[0, \varepsilon]$ and $i = 1, \ldots, N$, and that all L_i's are equal. Then, setting $\chi(0, S(0)) = \alpha_0$, we have that for $t \in [0, T_{j-1}]$,*

$$C \le \chi^2(t,y) \le D(1+y^2). \tag{50}$$

Moreover, the SDE (49) admits a unique strong solution.

The general treatment of Section 4 implies that the caplet price associated to (49) is

$$\begin{aligned}\mathbf{Cpl}(0, T_{j-1}, T_j, \tau_j, K) = \tau_j P(0, T_j) \sum_{i=1}^N \lambda_i \Bigg[\frac{F_j(0)}{2\sinh(-L_i)} \Big(e^{-L_i} \Phi\left(\bar{y}_i(T_{j-1}) + A_i(T_{j-1})\right) \\ - e^{L_i} \Phi\left(\bar{y}_i(T_{j-1}) - A_i(T_{j-1})\right) \Big) - K \Phi\big(\bar{y}_i(T_{j-1})\big) \Bigg].\end{aligned} \tag{51}$$

As anticipated, this caplet price leads to steep skews in the implied volatility curve. An example of the shape that can be reproduced is shown in Figure 4.

8 A second general class

Analytical tractability is often a key property for financial models. In general, the calibration to market option data can be extremely cumbersome and time consuming when resorting to numerical methods. This can be a serious drawback for a trader who has to price a new contract or an old one or, even worse, re-evaluate an entire book. In practice, therefore, having closed-form formulas for (plain-vanilla) options may translate into the actual possibility of applying the considered model concretely. This is the main reason why, in this article, we stick to analytically tractable models.

Explicit formulas for options, moreover, must be combined with a good fitting to market data. In this respect, models (3) and (9) are likely to be outperformed by (30). Traders

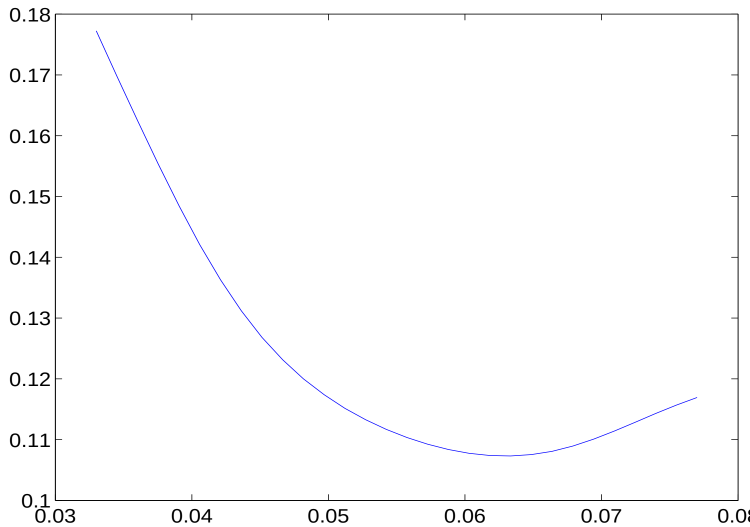

Figure 4: Caplet volatility curve implied by price (51), where we set, $T_{j-1} = 1$, $T_j = 1.5$, $\tau_j = 0.5$ $N = 2$, $(A_1(1), A_2(1)) = (0.01, 0.04)$, $(L_1, L_2) = (-0.056, -0.408)$, $(\lambda_1, \lambda_2) = (0.1, 0.9)$ and $F_j(0) = 0.055$.

love small calibration errors and naturally prefer models that best reproduce their market conditions. However, the goodness of a calibration is also measured, in a less direct way, by checking the evolution of the relevant implied volatility structures in the future, see Brigo and Mercurio (2001b, 2002a). In fact, unusual or unrealistic patterns for future volatilities may have a strong (negative) impact on the pricing, and especially hedging, of interest rates derivatives.

The future evolution of implied volatilities can be rapidly checked only in the presence of explicit future caplet prices, which is directly connected to the possibility of expressing analytically the transition density of the forward rate process. In this respect, models (3) and (9), with their formulas (6) and (11), are certainly preferable to (30), for which we only know the marginal density (31).

There is, therefore, the need for a FLM, which is capable of combining, at the same time, a good fitting with a complete tractability. This is why we introduce the following class of models.

Assume now that F_j can be expressed by the following (time-dependent) transformation of the Brownian motion W:

$$F_j(t) = h(t, W_t) \quad \text{for each } 0 \le t \le T_{j-1}, \tag{52}$$

where the function h satisfies:[13]

A1) h belongs to $C^{1,2}(\mathcal{D})$, with $\mathcal{D} := [0, T_{j-1}] \times I\!R$;

A2) $h(t, w) > 0$ for each $(t, w) \in \mathcal{D}$;

A3) for each $t > 0$, the function $h_t : I\!R \to I\!R^+$, $w \mapsto h_t(w) := h(t, w)$ has zero limit at minus infinity, $\lim_{w \to -\infty} h_t(w) = 0$, and is (strictly) increasing, i.e. $dh_t(w)/dw > 0$

[13] We actually consider different functions $h = h^j$ for different j's. We drop the superscript j to lighten the notation.

(equivalently, $\partial h(t,w)/\partial w > 0$), so that, for each $t > 0$, the function h_t is invertible and the inverse function h_t^{-1} is differentiable;

A4) $E^j\{h(T_{j-1}, W_{T_{j-1}})\}$ exists finite and $E^j\{h(T_{j-1}, W_{T_{j-1}})|\mathcal{F}_t\} = h(t, W_t)$, for each $0 \leq t \leq T_{j-1}$, so that F_j is indeed a martingale.

A simple example of function h that fulfills these requirements is the exponential $h(t,w) = a\exp(-b^2t/2 + bw)$, with $a, b > 0$, which leads to the lognormal formulation of the LIBOR forward model, see Brigo and Mercurio (2001b).

The purpose of this section is to design a general framework which preserves the analytical tractability of a geometric Brownian motion, but includes models that can produce realistic implied-volatility smiles. In the following, we show the advantages of our assumptions.

The related SDE

The SDE followed by the forward rate F_j is immediately derived by applying Ito's lemma:

$$\begin{aligned} dF_j(t) &= \left[\frac{\partial h}{\partial t}(t, W_t) + \frac{1}{2}\frac{\partial^2 h}{\partial w^2}(t, W_t)\right] dt + \frac{\partial h}{\partial w}(t, W_t)\, dW_t \\ &= \frac{\partial h}{\partial w}\left(t, h_t^{-1}(F_j(t))\right) dW_t, \\ &= \sigma(t, F_j(t))F_j(t)\, dW_t, \end{aligned} \tag{53}$$

with the obvious definition of the *local volatility* function $\sigma(\cdot,\cdot)$, and where the drift term is zero due to the last assumption on h and the Feynman-Kač theorem.[14] The process F_j is therefore a (one-dimensional) diffusion.

Marginal density

Denote by p_t the marginal density of $F_j(t)$, $t \leq T_{j-1}$. We have:

$$Q^j\{F_j(t) \leq x\} = Q^j\{h_t(W_t) \leq x\} = Q^j\{W_t \leq h_t^{-1}(x)\} = \Phi\left(\frac{h_t^{-1}(x)}{\sqrt{t}}\right). \tag{54}$$

Differentiating with respect to x yields the marginal density

$$p_t(x) = \frac{d}{dx}Q^j\{F_j(t) \leq x\} = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{1}{2t}[h_t^{-1}(x)]^2}\frac{d}{dx}h_t^{-1}(x). \tag{55}$$

Transition density

Given two any instants $t < T \leq T_{j-1}$, from definition (52) we get that the forward rate $F_j(T)$ conditional on $F_j(t)$ can be written as

$$F_j(T) = h\left(T, h_t^{-1}(F_j(t)) + W_T - W_t\right). \tag{56}$$

[14] This is obviously consistent with the martingale assumption on F_j.

Equation (56) allows us to derive the transition density of the process F_j. In fact, denoting by $p(t, y; T, x)$ the density of $F_j(T)$ conditional on $F_j(t) = y$ we have

$$\begin{aligned} Q^j\left\{F_j(T) \le x | F_j(t) = y\right\} &= Q^j\left\{h\left(T, h_t^{-1}(y) + W_T - W_t\right) \le x | F_j(t) = y\right\} \\ &= Q^j\left\{W_T - W_t \le h_T^{-1}(x) - h_t^{-1}(y) | F_j(t) = y\right\} \\ &= \Phi\left(\frac{h_T^{-1}(x) - h_t^{-1}(y)}{\sqrt{T-t}}\right), \end{aligned}$$

so that $p(t, y; T, x)$ is immediately obtained through differentiation with respect to x, namely

$$p(t, y; T, x) = \frac{d}{dx} Q^j\left\{F_j(T) \le x | F_j(t) = y\right\} = \frac{e^{-\frac{1}{2(T-t)}\left[h_T^{-1}(x) - h_t^{-1}(y)\right]^2}}{\sqrt{2\pi(T-t)}} \frac{d}{dx} h_T^{-1}(x). \tag{57}$$

Useful characterizations

We first notice that assumptions A3) and A4) imply the following.

Lemma 8.1. *For each $t < T_{j-1}$, the function h_t can be written in terms of $h_{T_{j-1}}^{-1}$ as*

$$h_t(w) = \int_0^{+\infty} \Phi\left(\frac{w - h_{T_{j-1}}^{-1}(z)}{\sqrt{T_{j-1} - t}}\right) dz. \tag{58}$$

Proof. From assumption A4), and by integration by parts, we get

$$\begin{aligned} h_t(w) &= \int_{-\infty}^{+\infty} h_{T_{j-1}}(x) \frac{1}{\sqrt{2\pi(T_{j-1} - t)}} e^{-\frac{(x-w)^2}{2(T_{j-1}-t)}} dx \\ &= \left[-h_{T_{j-1}}(x)\Phi\left(\frac{w - x}{\sqrt{T_{j-1} - t}}\right)\right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \frac{d}{dx} h_{T_{j-1}}(x)\Phi\left(\frac{w - x}{\sqrt{T_{j-1} - t}}\right) dx. \end{aligned}$$

The first term in the right-hand side of the last equality is zero since

$$\lim_{x \to -\infty} h_{T_{j-1}}(x)\Phi\left(\frac{w - x}{\sqrt{T_{j-1} - t}}\right) = 0$$

by assumption A3), whereas

$$\lim_{x \to +\infty} h_{T_{j-1}}(x)\Phi\left(\frac{w - x}{\sqrt{T_{j-1} - t}}\right) = 0$$

by assumption A4) and the fact that, for sufficiently large x,

$$\Phi\left(\frac{w - x}{\sqrt{T_{j-1} - t}}\right) \le \frac{\sqrt{T_{j-1} - t}}{x - w} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-w)^2}{2(T_{j-1}-t)}}.$$

Equality (58) is then obtained through the change of variable $z = h_{T_{j-1}}(x)$. □

Assuming we can swap integrals and derivatives, straightforward application of the derivation rule for a parameter-dependent integral leads to the following.

Corollary 8.2. *For each $t < T_{j-1}$, the derivative of function h_t can be written as*

$$\frac{d}{dw}h_t(w) = \int_0^{+\infty} \frac{\exp\Big\{-\frac{1}{2(T_{j-1}-t)}\big[w - h_{T_{j-1}}^{-1}(z)\big]^2\Big\}}{\sqrt{2\pi(T_{j-1}-t)}}\,dz. \tag{59}$$

Caplet pricing

The price at time t of the caplet resetting in T_{j-1}, paying in T_j and with strike K is given by

$$\begin{aligned}
&\mathbf{Cpl}(t, T_{j-1}, T_j, \tau_j, K) = \tau_j P(t, T_j) E^j\{[F_j(T_{j-1}) - K]^+ | \mathcal{F}_t\} \\
&= \tau_j P(t, T_j) \int_{-\infty}^{+\infty} \big[h_{T_{j-1}}\big(h_t^{-1}(F_j(t)) + w\big) - K\big]^+ \frac{1}{\sqrt{2\pi(T_{j-1}-t)}}\, e^{-\frac{w^2}{2(T_{j-1}-t)}}\,dw \\
&= \tau_j P(t, T_j) \Bigg[\int_{h_{T_{j-1}}^{-1}(K) - h_t^{-1}(F_j(t))}^{+\infty} \frac{h_{T_{j-1}}\big(h_t^{-1}(F_j(t)) + w\big)}{\sqrt{2\pi(T_{j-1}-t)}}\, e^{-\frac{w^2}{2(T_{j-1}-t)}}\,dw \\
&\quad - K \int_{h_{T_{j-1}}^{-1}(K) - h_t^{-1}(F_j(t))}^{+\infty} \frac{1}{\sqrt{2\pi(T_{j-1}-t)}}\, e^{-\frac{w^2}{2(T_{j-1}-t)}}\,dw\Bigg] \\
&= \tau_j P(t, T_j) \int_{h_{T_{j-1}}^{-1}(K) - h_t^{-1}(F_j(t))}^{+\infty} \frac{h_{T_{j-1}}\big(h_t^{-1}(F_j(t)) + w\big)}{\sqrt{2\pi(T_{j-1}-t)}}\, e^{-\frac{w^2}{2(T_{j-1}-t)}}\,dw \\
&\quad - K\tau_j P(t, T_j)\Phi\left(\frac{h_t^{-1}(F_j(t)) - h_{T_{j-1}}^{-1}(K)}{\sqrt{T_{j-1}-t}}\right),
\end{aligned} \tag{60}$$

where, in general, the final integral must be calculated numerically. In the following, however, we will consider an example where this integral can be calculated in an explicit fashion.

8.1 A particular case: a mixture of GBMs

As a particular case of the general dynamics (52), we consider a linear combination of N driftless and perfectly correlated geometric Brownian motions:

$$\begin{aligned}
F_j(t) &= h(t, W_t) \quad \text{for each } t \geq 0, \\
h(t, w) &= h_t(w) = \sum_{i=1}^N \psi_i\, e^{-\frac{1}{2}\beta_i^2 t + \beta_i w},
\end{aligned} \tag{61}$$

where $F_j(0)$, β_i's and ψ_i's are positive constants.

It is straightforward to show that this function fulfills our assumptions A1) to A4) and that the derivative of its inverse h_t^{-1} is

$$\frac{d}{dx}h_t^{-1}(x) = \frac{1}{\frac{d}{dw}h_t(h_t^{-1}(x))} = \frac{1}{\sum_{i=1}^N \psi_i\beta_i\, e^{-\frac{1}{2}\beta_i^2 t + \beta_i h_t^{-1}(x)}}.$$

The initial condition imposes that

$$\sum_{i=1}^{N} \psi_i = F_j(0),$$

so that setting, for each i, $\lambda_i := \psi_i / F_j(0)$, we can write F_j as a mixture of N (driftless) geometric Brownian motions starting at $F_j(0)$:

$$\begin{aligned} F_j(t) &= \sum_{i=1}^{N} \lambda_i Y_i(t) \\ dY_i(t) &= Y_i(t)\beta_i \, dW_t, \quad Y_i(0) = F_j(0). \end{aligned} \tag{62}$$

Straightforward application of Ito's lemma implies that

$$\begin{aligned} dF_j(t) &= \sum_{i=1}^{N} \lambda_i Y_i(t)\beta_i \, dW_t \\ &= \sum_{i=1}^{N} \psi_i \beta_i e^{-\frac{1}{2}\beta_i^2 t + \beta_i h_t^{-1}(F_j(t))} \, dW_t \\ &= \sigma(t, F_j(t)) F_j(t) \, dW_t, \end{aligned} \tag{63}$$

which is obviously consistent with (53). Notice we can also write

$$\begin{aligned} dF_j(t) &= F_j(t) \sum_{i=1}^{N} \Lambda_i(t, F_j(t)) \beta_i \, dW_t \\ \Lambda_i(t, z) &:= \frac{\psi_i \, e^{-\frac{1}{2}\beta_i^2 t + \beta_i h_t^{-1}(z)}}{\sum_{k=1}^{N} \psi_k \, e^{-\frac{1}{2}\beta_k^2 t + \beta_k h_t^{-1}(z)}}. \end{aligned}$$

This local volatility $\sigma(\cdot, \cdot)$ can thus be viewed as a stochastic weighted average of the basic volatilities β_i's, since the Λ_i's are positive and sum up to one.

Marginal and transition densities

From the general formulas (55) and (57), we get that the marginal and transition density functions of F_j are given respectively by

$$p_t(x) = \frac{e^{-\frac{1}{2t}\left(h_t^{-1}(x)\right)^2}}{\sqrt{2\pi t} \sum_{i=1}^{N} \psi_i \beta_i e^{-\frac{1}{2}\beta_i^2 t + \beta_i h_t^{-1}(x)}}, \tag{64}$$

and

$$p(t, y; T, x) = \frac{e^{-\frac{1}{2(T-t)}\left[h_T^{-1}(x) - h_t^{-1}(y)\right]^2}}{\sqrt{2\pi(T-t)} \sum_{i=1}^{N} \psi_i \beta_i \, e^{-\frac{1}{2}\beta_i^2 T + \beta_i h_T^{-1}(x)}}. \tag{65}$$

Caplet pricing

We calculate the integral in the last member of (60) and obtain

$$\begin{aligned}
&\int_{h_{T_{j-1}}^{-1}(K)-h_t^{-1}(F_j(t))}^{+\infty} \frac{h_{T_{j-1}}\left(h_t^{-1}(F_j(t))+w\right)}{\sqrt{2\pi(T_{j-1}-t)}}\, e^{-\frac{w^2}{2(T_{j-1}-t)}}\,dw \\
&= \int_{h_{T_{j-1}}^{-1}(K)-h_t^{-1}(F_j(t))}^{+\infty} \sum_{i=1}^{N} \psi_i\, e^{-\frac{1}{2}\beta_i^2 T_{j-1}+\beta_i h_t^{-1}(F_j(t))+\beta_i w} \frac{1}{\sqrt{2\pi(T_{j-1}-t)}}\, e^{-\frac{w^2}{2(T_{j-1}-t)}}\,dw \\
&= \sum_{i=1}^{N} \psi_i\, e^{-\frac{1}{2}\beta_i^2 T_{j-1}+\beta_i h_t^{-1}(F_j(t))} \int_{h_{T_{j-1}}^{-1}(K)-h_t^{-1}(F_j(t))}^{+\infty} \frac{1}{\sqrt{2\pi(T_{j-1}-t)}}\, e^{\beta_i w-\frac{w^2}{2(T_{j-1}-t)}}\,dw \\
&= \sum_{i=1}^{N} \psi_i\, e^{-\frac{1}{2}\beta_i^2 t+\beta_i h_t^{-1}(F_j(t))} \int_{h_{T_{j-1}}^{-1}(K)-h_t^{-1}(F_j(t))}^{+\infty} \frac{1}{\sqrt{2\pi(T_{j-1}-t)}}\, e^{-\frac{[w-\beta_i(T_{j-1}-t)]^2}{2(T_{j-1}-t)}}\,dw
\end{aligned}$$

which immediately leads to the caplet price

$$\begin{aligned}
&\mathbf{Cpl}(t, T_{j-1}, T_j, \tau_j, K) \\
&= \tau_j P(t,T_j) \sum_{i=1}^{N} \psi_i\, e^{-\frac{1}{2}\beta_i^2 t+\beta_i h_t^{-1}(F_j(t))} \Phi\left(\frac{\beta_i(T_{j-1}-t) - h_{T_{j-1}}^{-1}(K) + h_t^{-1}(F_j(t))}{\sqrt{T_{j-1}-t}}\right) \\
&\quad - K\tau_j P(t,T_j)\Phi\left(\frac{h_t^{-1}(F_j(t)) - h_{T_{j-1}}^{-1}(K)}{\sqrt{T_{j-1}-t}}\right).
\end{aligned} \tag{66}$$

In particular, the caplet price at time $t = 0$ reduces to

$$\mathbf{Cpl}(0, T_{j-1}, T_j, \tau_j, K) = \tau_j P(0,T_j)\left[\sum_{i=1}^{N} \psi_i\, \Phi\left(\frac{\beta_i T_{j-1} - h_{T_{j-1}}^{-1}(K)}{\sqrt{T_{j-1}}}\right) - K\Phi\left(-\frac{h_{T_{j-1}}^{-1}(K)}{\sqrt{T_{j-1}}}\right)\right]. \tag{67}$$

The implied volatility curves obtained from the caplet prices (67) typically show weird patterns (e.g. are increasing and concave in the strike),[15] which renders our GBM mixture model hardly suitable for calibration to market data. However, we can resort to a slightly more general model, which is described in the following.

8.2 An extension of the GBM mixture model allowing for implied volatility skews

It turns out that dealing with (strictly) positive combinators ψ is too restrictive as far the shape of the implied volatility surface is concerned. We now relax this assumption and allow for negative combinators, without possibly losing the analytical tractability of the initial model.

One of the key features in the definition (61) is that, for each fixed t, the function $h(t,\cdot)$ is increasing and invertible, property which is easily lost if some ψ_i is negative. In order to

[15] This behaviour is also typical of the shifted-lognormal model (3) when the shift parameter α is positive. This analogy is not surprising. Notice in fact that the shifted-lognormal model is a particular case of GBM mixture model (take $N = 2$, $\beta_1 = 0$ and $\beta_2 > 0$).

preserve the desired behaviour of h, we assume that each "volatility" parameter β_i has the same sign of the corresponding combinator ψ_i. Summarizing

$$\text{for each } i = 1, \ldots, N: \ \psi_i, \, \beta_i \in I\!R, \ \text{but } \operatorname{sign}(\psi_i \beta_i) = 1. \tag{68}$$

Trivially, under (68) the function h_t is still differentiable, increasing and invertible. This basically means that formulas (64), (65) and (66) still hold true, with no modification. However, we have lost something: the positivity of function h, and hence of the price process F_j. This is a rather undesirable feature, which we must accommodate somehow.

We proceed as follows. We fix $\delta > 0$ and set $\bar{w} := h^{-1}_{T_{j-1}}(0) + \delta$, which is well and uniquely defined (we omit the dependence on j for brevity), and $\varepsilon := h_{T_{j-1}}(\bar{w})$. We also set

$$\begin{aligned} \alpha &:= \frac{dh_{T_{j-1}}}{dw}(\bar{w}), \\ \beta &:= \frac{d^2 h_{T_{j-1}}}{dw^2}(\bar{w}), \end{aligned} \tag{69}$$

and assume that $\beta < \alpha^2/\varepsilon$. If we define:

$$f(w) := a\, e^{bw + cw^2},$$

where

$$\begin{aligned} a &:= \varepsilon\, e^{-b\bar{w} - c\bar{w}^2}, \\ b &:= \frac{\alpha}{\varepsilon} - \bar{w}\left(\frac{\beta}{\varepsilon} - \frac{\alpha^2}{\varepsilon^2}\right), \\ c &:= \frac{\beta}{2\varepsilon} - \frac{\alpha^2}{2\varepsilon^2} < 0, \end{aligned} \tag{70}$$

we have that f satisfies

$$\begin{aligned} & f(w) > 0 \text{ for each } w \in I\!R, \quad f(w) \in C^2(I\!R), \\ & \frac{df}{dw}(w) > 0 \text{ for each } w < \bar{w}, \quad \lim_{w \to -\infty} f(w) = 0, \\ & f(\bar{w}) = \varepsilon, \quad \frac{df}{dw}(\bar{w}) = \alpha, \quad \frac{d^2 f}{dw^2}(\bar{w}) = \beta. \end{aligned}$$

We can then define a new function $h_{T_{j-1}}$, which we shall denote by $\bar{h}_{T_{j-1}}$, such that

$$\bar{h}_{T_{j-1}}(w) := \begin{cases} h_{T_{j-1}}(w) & \text{if } w \geq \bar{w}, \\ f(w) & \text{if } w < \bar{w}. \end{cases} \tag{71}$$

We can immediately notice that this function $\bar{h}_{T_{j-1}}$ possesses all the advantages of function $h_{T_{j-1}}$ in (61), including positivity. Moreover, provided we choose strikes $K \geq \varepsilon$, prices of T_j-maturity caplets under $\bar{h}_{T_{j-1}}$ coincide with the corresponding option prices (66) under $h_{T_{j-1}}$. For strikes lower than ε we can still derive a closed form formula for a caplet price. However, if we take δ in such a way that ε is the smallest quoted strike, the caplet price (66) is all that really matters.

We have thus solved the positivity issue at expiry T_{j-1}. The price we paid is that $\bar{h}_{T_{j-1}}(W_{T_{j-1}})$ does not have the same expected value as $h_{T_{j-1}}(W_{T_{j-1}})$, which essentially means we are violating no-arbitrage. However, we can replace the coefficients ψ_i's, implicit in (69), (70) and (71), with new (and smaller) $\bar{\psi}_i$'s such that:[16]

$$E[\bar{h}_{T_{j-1}}(W_{T_{j-1}})] = F_j(0),$$

and we are basically done. Two examples of volatility curves implied by the option price (66), under (68), are shown in Figure 5.

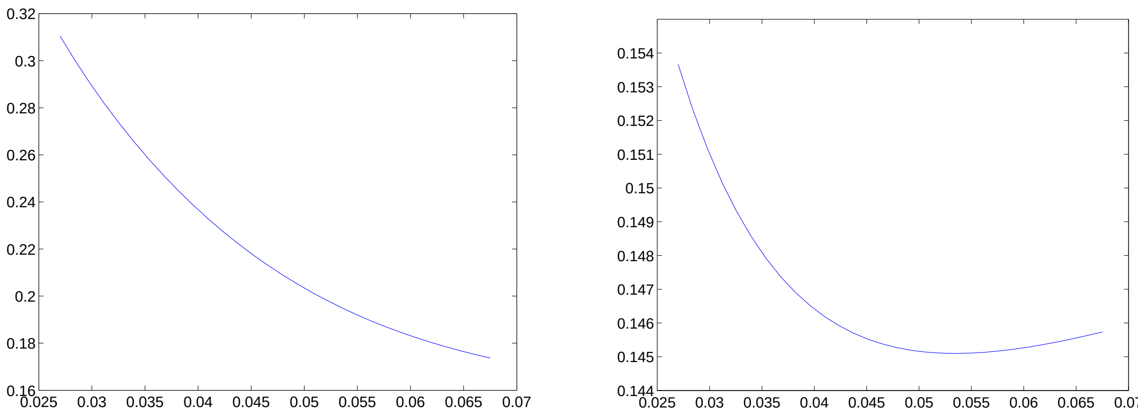

Figure 5: Implied volatility curves for the caplet price (66), where $F_j(0) = 0.045$, $T_{j-1} = 1$, $(\psi_1, \psi_2, \psi_3) = \{0.02475, -0.01125, 0.0315\}$ and $\bar{w} = -2$, so that $(\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3) = \{0.02474376, -0.01124716, 0.031492056\}$ and $\varepsilon = 0.02368$. Left: $(\beta_1, \beta_2, \beta_3) = \{0.2, -0.4, 0.0155\}$. Right: $(\beta_1, \beta_2, \beta_3) = \{0.2, -0.1, 0.0155\}$.

One is then tempted to extend the above procedure to every time instant $t < T_{j-1}$, leading to (positive) functions $\bar{h}_t(\cdot)$. However, doing this we would miss a crucial feature: the forward rate process must be a martingale under Q^j. We therefore proceed as follows.

To meet the no-arbitrage requirement, we define

$$F_j(t) = \bar{h}_t(W_t)$$
$$\bar{h}_t(w) = E^j[\bar{h}_{T_{j-1}}(W_{T_{j-1}})|W_t = w].$$

This conditional expectation can be calculated by writing $W_{T_{j-1}} = W_t + (W_{T_{j-1}} - W_t)$ and remembering that $W_{T_{j-1}} - W_t$ is independent of W_t. By completing the squares in the

[16]We can indeed prove the existence of such $\bar{\psi}_i$'s.

exponents and setting $\Delta t := T_{j-1} - t$, we obtain:

$$
\begin{aligned}
&E^j[\bar{h}_{T_{j-1}}(W_{T_{j-1}})|W_t = w] = \int_{-\infty}^{+\infty} \bar{h}_{T_{j-1}}(w+x)\frac{1}{\sqrt{2\pi\Delta t}}\, e^{-\frac{x^2}{2\Delta t}}\,dx \\
&= \int_{-\infty}^{\bar{w}-w} a\frac{e^{b(w+x)+c(w+x)^2-\frac{x^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}}\,dx + \int_{\bar{w}-w}^{+\infty} \frac{1}{\sqrt{2\pi\Delta t}}\sum_{i=1}^{N}\bar{\psi}_i\, e^{-\frac{1}{2}\beta_i^2 T_{j-1}+\beta_i(w+x)-\frac{x^2}{2\Delta t}}\,dx \\
&= a\,e^{bw+cw^2}\int_{-\infty}^{\bar{w}-w}\frac{e^{bx+2cwx+cx^2-\frac{x^2}{2\Delta t}}}{\sqrt{2\pi\Delta t}}\,dx + \sum_{i=1}^{N}\bar{\psi}_i\, e^{-\frac{1}{2}\beta_i^2 T_{j-1}+\beta_i w}\int_{\bar{w}-w}^{+\infty}\frac{1}{\sqrt{2\pi\Delta t}}\, e^{\beta_i x-\frac{x^2}{2\Delta t}}\,dx \\
&= a\,e^{bw+cw^2+\frac{\Delta t(b+2cw)^2}{2(1-2c\Delta t)}}\int_{-\infty}^{\bar{w}-w}\frac{1}{\sqrt{2\pi\Delta t}}\, e^{-\frac{1-2c\Delta t}{2\Delta t}\left[x-\frac{(b+2cw)\Delta t}{1-2c\Delta t}\right]^2}\,dx \\
&\quad + \sum_{i=1}^{N}\bar{\psi}_i\, e^{-\frac{1}{2}\beta_i^2 t+\beta_i w}\int_{\bar{w}-w}^{+\infty}\frac{1}{\sqrt{2\pi\Delta t}}\, e^{-\frac{1}{2\Delta t}(x-\beta_i\Delta t)^2}\,dx \\
&= \frac{a\,e^{\frac{2bw+2cw^2+b^2\Delta t}{2(1-2c\Delta t)}}}{\sqrt{1-2c\Delta t}}\Phi\left(\frac{\bar{w}(1-2c\Delta t)-w-b\Delta t}{\sqrt{(1-2c\Delta t)\Delta t}}\right) + \sum_{i=1}^{N}\bar{\psi}_i\, e^{-\frac{1}{2}\beta_i^2 t+\beta_i w}\Phi\left(-\frac{\bar{w}-w-\beta_i\Delta t}{\sqrt{\Delta t}}\right)
\end{aligned}
$$

We thus have:

$$
\begin{aligned}
\bar{h}_t(w) = {}& \frac{a\,e^{\frac{2bw+2cw^2+b^2(T_{j-1}-t)}{2(1-2c(T_{j-1}-t))}}}{\sqrt{1-2c(T_{j-1}-t)}}\,\Phi\left(\frac{\bar{w}(1-2c(T_{j-1}-t))-w-b(T_{j-1}-t)}{\sqrt{(1-2c(T_{j-1}-t))(T_{j-1}-t)}}\right) \\
&+\sum_{i=1}^{N}\bar{\psi}_i\, e^{-\frac{1}{2}\beta_i^2 t+\beta_i w}\Phi\left(\frac{w-\bar{w}+\beta_i(T_{j-1}-t)}{\sqrt{T_{j-1}-t}}\right)
\end{aligned}
\tag{72}
$$

The function $w \mapsto \bar{h}_t(w)$ possesses all the advantages of the original function h_t:

- $\bar{h}_t > 0$ for each $(t, w) \in \mathcal{D}$ since $\bar{h}_{T_{j-1}}$ is positive and the sign is preserved when taking expectation;
- for each $t > 0$, the function $\bar{h}_t$ is differentiable in w and $d\bar{h}_t(w)/dw > 0$; in fact,

  $$\frac{d\bar{h}_t(w)}{dw} = \int_{-\infty}^{+\infty}\frac{d}{dw}\bar{h}_{T_{j-1}}(w+x)\frac{1}{\sqrt{2\pi(T_{j-1}-t)}}\, e^{-\frac{x^2}{2(T_{j-1}-t)}}\,dx > 0,$$

  since $d\bar{h}_{T_{j-1}}(z)/dz > 0$ for each z. Accordingly, $\bar{h}_t$ is (strictly) increasing;
- for each $t > 0$, the function h_t is invertible;
- for each $0 < s < t$, the transition density $p(s, y; t, x)$ can be obtained through numerical inversion of $\bar{h}_t$.

Remark 8.3. *The above construction procedure is quite general and can be used for any initial function $h_{T_{j-1}}$ being increasing.*

8.3 A general dynamics à la Dupire (1994)

We now consider a second FLM belonging to the general class (52). This model has the advantage of exactly retrieving the caplet volatility smile for the associated forward rate. With respect to the previous GBM-mixture model, the relevant formulas are less explicit, requiring numerical integrations. However, the calibration is automatic and calculations are still fast and efficient.[17]

Assume now that caplet prices with reset and maturity dates in T_{j-1} and T_j, respectively, are available (in the market) for a continuum of strikes. Precisely, denoting by $C_j(K)$ the caplet price with strike K, the following no-arbitrage conditions are assumed to hold:

B1) $C_j \in C^2((0,+\infty))$;

B2) $\lim_{x\to 0^+} C_j(x) = \tau_j P(0,T_j)F_j(0)$ and $\lim_{x\to +\infty} C_j(x) = 0$;

B3) $\lim_{x\to 0^+} \frac{dC_j}{dx}(x) = -\tau_j P(0,T_j)$ and $\lim_{x\to +\infty} x\frac{dC_j}{dx}(x) = 0$;

B4) $\frac{d^2C_j}{dx^2}(x) > 0$ for each $x > 0$, implying $-\tau_j P(0,T_j) < \frac{dC_j}{dx}(x) < 0$ for each $x > 0$.

Proposition 8.4. *The function $h_{T_{j-1}}$ that is consistent with the given caplet prices is implicitly defined by*

$$h^{-1}_{T_{j-1}}(x) = -\sqrt{T_{j-1}}\,\Phi^{-1}\left(-\frac{\frac{dC_j}{dx}(x)}{\tau_j P(0,T_j)}\right), \quad x > 0, \tag{73}$$

which is well defined due to the assumptions on C_j.

Proof. Following Breeden and Litzenberger (1978) and applying (54), we get that

$$\begin{aligned}\frac{\partial}{\partial K}\mathbf{Cpl}(0,T_{j-1},T_j,\tau_j,K) &= \tau_j P(0,T_j)\left[Q^j\{F_j(T_{j-1})\le K\} - 1\right]\\ &= \tau_j P(0,T_j)\left[\Phi\left(\frac{h^{-1}_{T_{j-1}}(K)}{\sqrt{T_{j-1}}}\right) - 1\right]\\ &= -\tau_j P(0,T_j)\Phi\left(-\frac{h^{-1}_{T_{j-1}}(K)}{\sqrt{T_{j-1}}}\right).\end{aligned}$$

Imposing an exact match of the market caplet prices, we must have

$$\frac{dC_j}{dK}(K) = -\tau_j P(0,T_j)\Phi\left(-\frac{h^{-1}_{T_{j-1}}(K)}{\sqrt{T_{j-1}}}\right),$$

which immediately leads to (73) through inversion of the standard normal cumulative distribution function, given the bounds on the derivative $dC_j(x)/dx$. □

[17] The treatment in this section is similar to that of Balland and Hughston (2000) who proved the existence of a LIBOR forward model that depends on some given Markovian factors and exactly calibrates the market caplet volatility curves. However, our formulation is more explicit and our assumptions lead to more tractable models.

Corollary 8.5. *The function $h_{T_{j-1}}$ can be explicitly written as*

$$h_{T_{j-1}}(w) = \left(\frac{dC_j}{dx}\right)^{-1}\left(-\tau_j P(0,T_j)\Phi\left(-\frac{w}{\sqrt{T_{j-1}}}\right)\right), \quad w \in I\!R, \tag{74}$$

with $\left(\frac{dC_j}{dx}\right)^{-1}$ denoting the inverse function of the first derivative of C_j.

Proof. We take $x = h_{T_{j-1}}(w)$ in (73) obtaining

$$-\frac{w}{\sqrt{T_{j-1}}} = \Phi^{-1}\left(-\frac{\frac{dC_j}{dx}(h_{T_{j-1}}(w))}{\tau_j P(0,T_j)}\right),$$

and apply function Φ to both sides. □

Corollary 8.6. *The function $h_{T_{j-1}}$ is strictly positive, differentiable, increasing and with zero limit at minus infinity,*

$$\lim_{w\to-\infty} h_{T_{j-1}}(w) = 0. \tag{75}$$

Moreover, the Q^j-expectation of $h_{T_{j-1}}(W_{T_{j-1}})$ is finite and equal to $F_j(0)$.

Proof. Since (73) holds for $x > 0$ (the domain of $h^{-1}_{T_{j-1}}$ is the range of $h_{T_{j-1}}$), $h_{T_{j-1}}$ is (strictly) positive. Moreover,

$$\frac{d}{dw}h_{T_{j-1}}(w) = \frac{1}{\frac{d^2C_j}{dx^2}(h_{T_{j-1}}(w))}\,\frac{\tau_j P(0,T_j)}{\sqrt{2\pi T_{j-1}}}\,e^{-\frac{w^2}{2T_{j-1}}},$$

which is (strictly) positive due to B4). The limit in (75) then follows from (74) and assumptions B3) and B4). Finally, remembering (54),

$$\begin{aligned}
E^j[h_{T_{j-1}}(W_{T_{j-1}})] &= -\lim_{x\to+\infty} x\,Q^j\{F_j(T_{j-1}) \geq x\} + \int_0^{+\infty} Q^j\{F_j(T_{j-1}) \geq x\}dx \\
&= -\lim_{x\to+\infty} x\,\Phi\left(-\frac{h^{-1}_{T_{j-1}}(x)}{\sqrt{T_{j-1}}}\right) + \int_0^{+\infty}\Phi\left(-\frac{h^{-1}_{T_{j-1}}(x)}{\sqrt{T_{j-1}}}\right)dx \\
&= \frac{1}{\tau_j P(0,T_j)}\lim_{x\to+\infty} x\,\frac{dC_j}{dx}(x) - \frac{1}{\tau_j P(0,T_j)}\int_0^{+\infty}\frac{dC_j}{dx}(x) \\
&= 0 - \frac{1}{\tau_j P(0,T_j)}\left[\lim_{x\to+\infty} C_j(x) - \lim_{x\to 0^+} C_j(x)\right] \\
&= F_j(0),
\end{aligned}$$

where we have used (73) and assumptions B2) and B3). □

From a computational point of view, the value of function $h_{T_{j-1}}$ in any point w can be obtained either by means of the explicit definition (74) or by numerically solving the equation

$$w - h^{-1}_{T_{j-1}}(x) = 0$$

in the variable x.[18]

[18]Several algorithms are available for solving nonlinear equations. In this specific case, the solution search is extremely efficient since the nonlinear function is monotone and with an analytical gradient.

Proposition 8.7. *The value of the forward rate F_j at any time $t < T_{j-1}$ that is consistent with $h_{T_{j-1}}$ is*

$$F_j(t) = h_t(W_t), \tag{76}$$

where

$$\begin{aligned} h_t(w) &= E^j[h_{T_{j-1}}(W_{T_{j-1}})|W_t = w] \\ &= \int_0^{+\infty} \Phi\left(\frac{w + \sqrt{T_{j-1}}\,\Phi^{-1}\big(-\frac{1}{\tau_j P(0,T_j)}\frac{dC_j}{dz}(z)\big)}{\sqrt{T_{j-1}-t}}\right) dz. \end{aligned} \tag{77}$$

Proof. Since the expectation of $h_{T_{j-1}}(W_{T_{j-1}})$ under Q^j is finite, and equal to $F_j(0)$, the first equation in (77) defines the unique $(Q^j, \mathcal{F}_t)$-martingale with value $h_{T_{j-1}}(W_{T_{j-1}})$ at time T_{j-1} due to the Markov property of Brownian motion. The second equation in (77) then follows from (58) and (73).[19] □

From (77) we immediately see that h_t is positive for each $t < T_{j-1}$. Assuming twice differentiability for h_t and the possibility of swapping integrals and derivatives, the forward rate dynamics is then given by the following.

Corollary 8.8. *The (driftless) dynamics of F_j that is consistent with the forward rate value (77) is*

$$\begin{aligned} dF_j(t) &= \frac{\partial h_t}{\partial w}\big(h_t^{-1}(F_j(t))\big)\,dW_t \\ &= \int_0^{+\infty} \frac{\exp\Big\{-\frac{1}{2(T_{j-1}-t)}\big[h_t^{-1}(F_j(t)) + \sqrt{T_{j-1}}\,\Phi^{-1}\big(-\frac{1}{\tau_j P(0,T_j)}\frac{dC_j}{dz}(z)\big)\big]^2\Big\}}{\sqrt{2\pi(T_{j-1}-t)}}\,dz\,dW_t. \end{aligned} \tag{78}$$

Proof. The result follows from the differentiability assumption on h_t and from (59) with (73). □

In order to check the evolution of the implied volatility curves produced by the forward rate process in the future, we need to analytically price caplets at any future time $0 < t < T_{j-1}$. From the general pricing formula (60), we obtain the following.

Proposition 8.9. *The price at time $t < T_{j-1}$ of the caplet resetting in T_{j-1}, paying in T_j and with strike K is given by*

$$\begin{aligned} &\mathbf{Cpl}(t, T_{j-1}, T_j, \tau_j, K) \\ &\quad = \tau_j P(t,T_j)\left[F_j(t) - K + \int_0^K \Phi\left(\frac{h_{T_{j-1}}^{-1}(z) - h_t^{-1}(F_j(t))}{\sqrt{T_{j-1}-t}}\right) dz\right] \\ &\quad = \tau_j P(t,T_j)\left[F_j(t) - \int_0^K \Phi\left(\frac{\sqrt{T_{j-1}}\,\Phi^{-1}\big(-\frac{1}{\tau_j P(0,T_j)}\frac{dC_j}{dz}(z)\big) + h_t^{-1}(F_j(t))}{\sqrt{T_{j-1}-t}}\right) dz\right] \end{aligned} \tag{79}$$

and can easily be calculated with numerical methods.

[19] We can apply (58) since $\lim_{w\to-\infty} h_{T_{j-1}}(w) = 0$ and $E^j\{h_{T_{j-1}}(W_{T_{j-1}})\}$ exists finite.

Proof. The final integral in (60) can be rewritten as follows:

$$
\begin{aligned}
\cdots &= F_j(t) - \int_{-\infty}^{h_{T_{j-1}}^{-1}(K)-h_t^{-1}(F_j(t))} \frac{h_{T_{j-1}}\left(h_t^{-1}(F_j(t))+w\right)}{\sqrt{2\pi(T_{j-1}-t)}}\, e^{-\frac{w^2}{2(T_{j-1}-t)}}\, dw \\
&= F_j(t) - K\Phi\left(\frac{h_{T_{j-1}}^{-1}(K)-h_t^{-1}(F_j(t))}{\sqrt{T_{j-1}-t}}\right) \\
&\quad + \int_{-\infty}^{h_{T_{j-1}}^{-1}(K)-h_t^{-1}(F_j(t))} \frac{d}{dw} h_{T_{j-1}}\left(h_t^{-1}(F_j(t))+w\right)\Phi\left(\frac{w}{\sqrt{T_{j-1}-t}}\right) dw \\
&= F_j(t) - K\Phi\left(\frac{h_{T_{j-1}}^{-1}(K)-h_t^{-1}(F_j(t))}{\sqrt{T_{j-1}-t}}\right) + \int_0^K \Phi\left(\frac{h_{T_{j-1}}^{-1}(z)-h_t^{-1}(F_j(t))}{\sqrt{T_{j-1}-t}}\right) dz,
\end{aligned}
$$

where we have used the definition of $F_j(t)$ (first equality), the integration by parts (second equality) and the change of variable $z = h_{T_{j-1}}(h_t^{-1}(F_j(t))+w)$ (third equality).

Putting pieces together, we finally obtain the first equality in (79), whereas the second follows from (73). □

Remark 8.10. *Propositions 8.4 and 8.7 can be equivalently restated for a general pricing function C_j, which does not necessarily match all market quotes. In fact, we can assume a flexible parametric distribution for $F_j(T_{j-1}) = h_{T_{j-1}}(W_{T_{j-1}})$, thus avoiding the classic interpolation issues. For instance, we can consider a mixture of lognormal densities as in (40), where we set $t = T_{j-1}$ and take constant coefficients,*

$$
p_{T_{j-1}}(x) = \sum_{i=1}^{N} \lambda_i \frac{1}{x\sigma_i\sqrt{2\pi T_{j-1}}} \exp\left\{-\frac{1}{2\sigma_i^2 T_{j-1}}\left[\ln\frac{x}{F_j(0)} - \mu_i T_{j-1} + \tfrac{1}{2}\sigma_i^2 T_{j-1}\right]^2\right\},
$$

with $\sum_{i=1}^{N} \lambda_i\, e^{\mu_i T_{j-1}} = 1$. In this case, we have

$$
C_j(K) = \tau_j P(0,T_j) \sum_{i=1}^{N} \lambda_i e^{\mu_i T_{j-1}} \mathrm{Bl}\left(K\, e^{-\mu_i T_{j-1}}, F_j(0), \sigma_i\sqrt{T_{j-1}}\right),
$$

from which (73) and (76) can be derived.

We can then compare the resulting forward rate dynamics (78) with (42) under (43). It is evident that the latter has a more explicit formulation involving no numerical procedure (besides the standard ones for the computation of transcendent functions). However, the former has several advantages: it allows for i) less severe restrictions on the model parameters to ensure that the local volatility is well defined, ii) constant coefficients, and hence for a more parsimonious parameterization, iii) a rapid calculation of future caplet prices, and hence for a quick check of the future evolution of the caplet volatility structure.

9 An Example of Calibration to Market Data

We here test the fitting capability of the extended model (37) based on interest rates volatility data. Precisely, we use the caplet volatilities that are stripped from the quoted in-the-money and out-of-the-money Euro cap volatilities as of November 14th, 2000. We focus on the volatilities of the two-year caplets with the underlying LIBOR rate resetting at 1.5 years. The underlying forward rate is 5.32%, the considered strikes are 4%, 4.25%, 4.5%, 4.75%, 5%, 5.25%, 5.5%, 5.75%, 6%, 6.25%, 6.5% and the associated (mid) volatilities are 15.22%, 15.14%, 15.10%, 15.08%, 15.09%, 15.12%, 15.17%, 15.28%, 15.40%, 15.52%, 15.69%.

Setting $N = 2$, $v_i := V_i(1.5)$, $i = 1, 2$, and $\lambda_2 = 1 - \lambda_1$, we looked for the admissible values of λ_1, v_1, v_2 and α minimizing the squared percentage difference between model and market (mid) prices, with α satisfying the constraint $\alpha < K$ for each traded strike K.

We obtained $\lambda_1 = 0.2412$, $\lambda_2 = 0.7588$, $v_1 = 0.1527$, $v_2 = 0.2381$ and $\alpha = 0.0078$. The resulting implied volatilities are plotted in Figure 6, where they are compared with the market mid volatilities.

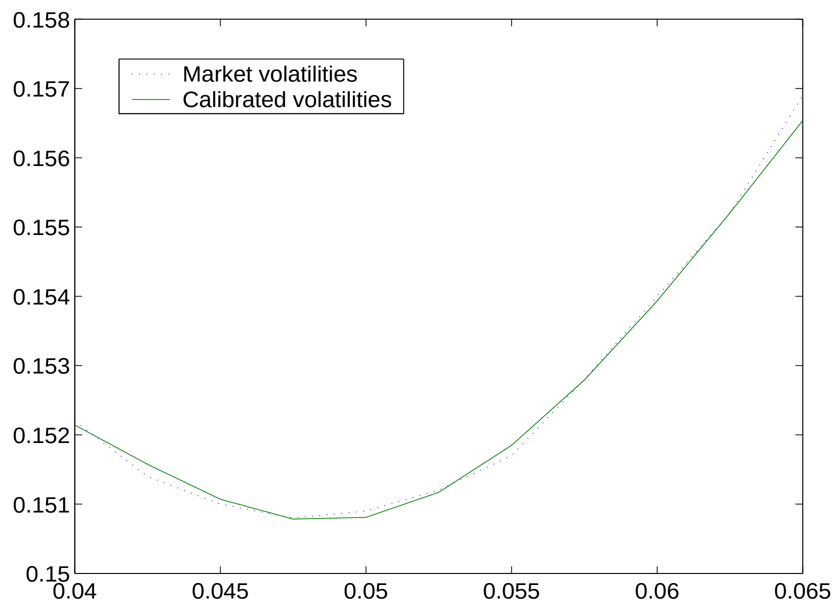

Figure 6: Plots of the calibrated volatilities vs the market mid volatilities.

Remark 9.1. *In this example we could obtain a satisfactory fitting to the considered market data already with a mixture of two densities. Indeed, the fact that the implied volatility smile is almost flat helped in achieving such a calibration result. However, in case one needs to reproduce steeper curves, we remind that models (42) and (49) can be more suitable for the purpose.*

10 Conclusions

We proposed several dynamics alternative to a geometric Brownian motion for modeling forward rates, under their canonical measure, in a LIBOR market model setup.

Our alternative dynamics are all analytically tractable in that they lead to closed form formulas for caplet prices.

The implied caplet volatility curves display typical market shapes. They range from the smile-shaped curve implied by a mixture of lognormal densities to the steep skew-shaped curve in case of a mixture based on hyperbolic-sine processes. The virtually unlimited number of parameters in our models, can indeed render the calibration to real market data extremely accurate in most cases.

Two are the main issues needing further investigation: i) the analysis of the evolution of the caplet volatility curves implied in the future by our models; ii) the stability in time of the model parameters. These are, indeed, the by-now classical problematic features one has to face when dealing with local-volatility models like ours.

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