A Vega-Gamma Relationship for European-Style or Barrier Options in the Black-Scholes Model

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Abstract

In this document we derive some fundamental relationships between the Greeks of a barrier option under the Black-Scholes model. A European-style option can be considered as a limit case. Besides the classical Black and Scholes P.D.E., a barrier option price is shown to also satisfy: i) a time homogeneity; ii) a scale invariance of time; iii) an interest rates symmetry; iv) a Vega-Gamma relationship.

Notation

t: the running time.
S_t: the price at time t in domestic currency of one unit of foreign currency.
r: the (constant) domestic instantaneous risk-free rate.
q: the (constant) foreign instantaneous risk-free rate.
σ: the exchange rate (constant) percentage volatility.
T: the maturity of a given derivative.
τ: the time-to-maturity of this given derivative, namely $\tau = T - t$.
V_T: the derivative's payoff at time T.
$V(t, S_t, r, q, \sigma, \tau)$: no-arbitrage price at time t of a given derivative security with time-to-maturity τ, when the underlying exchange rate is S, the domestic instantaneous risk-free rate is r, the foreign instantaneous risk-free rate is q and the exchange rate percentage volatility is σ. A shorthand notation for it is V_t.
Q^d: the domestic risk-neutral measure.
E^d: expectation under Q^d.
$\mathcal{F}_t$: the σ-algebra generated by S up to time t.

1.1 Assumptions

The exchange rate S is assumed to evolve under the domestic risk-neutral measure Q^d according to:

$$dS_t = S_t[(r - q)\,dt + \sigma\,dW_t]$$

where W is a standard Brownian motion under Q^d.

We are given a derivative security with payoff V_T at time T. This derivative is either a European-style option or a (single) barrier option. We actually restrict our analysis to "out" barrier options, which become worthless if the barrier is hit prior to maturity. In fact, a European-style payoff can be obtained by letting the barrier level either go to zero or to infinity, whereas an "in" barrier option is simply the difference between the corresponding European-style and "out" barrier options.

1.2 The pricing of a Barrier Option

Standard theory in derivatives pricing implies that the no-arbitrage price at time t of the payoff V_T at time T is

$$V_t = e^{-r\tau}\, E^d[V_T|\mathcal{F}_t] \tag{1}$$

Set $\nu := r - q - \frac{1}{2}\sigma^2$ and

$$X_t := \nu t + \sigma W_t$$
$$M_t^X := \max_{u\in[0,t]}(\omega X_u)$$

where $\omega = 1$ for maximum and $\omega = -1$ for minimum. Then, see for instance Musiela and Rutkowski (1998), for $\omega y \geq \max(0, \omega x)$,

$$Q^d\big\{X_T \in dx, M_T^X \leq \omega y\big\} = \omega\left[\frac{1}{\sigma\sqrt{2\pi T}}\, e^{-\frac{1}{2}\left(\frac{x-\nu T}{\sigma\sqrt{T}}\right)^2} - e^{2\frac{\nu y}{\sigma^2}}\frac{1}{\sigma\sqrt{2\pi T}}\, e^{-\frac{1}{2}\left(\frac{x-2y-\nu T}{\sigma\sqrt{T}}\right)^2}\right]$$

Assuming that the option payoff, if the barrier is not hit prior to maturity, is a function $h(S_T)$ of the underlying at maturity, the barrier option price at time 0 can be written as

$$V_0 = e^{-rT}\int_{z\in A} h(e^z) f(z; S_0, r, q, \sigma, T)\, dz$$

where neither h nor $A \subset I\!\!R$ depend on S_0, r, q, σ, T and

$$f(z; S, r, q, \sigma, T) = \omega\left[\frac{1}{\sigma\sqrt{2\pi T}}\, e^{-\frac{1}{2}\left(\frac{z-\ln S-\nu T}{\sigma\sqrt{T}}\right)^2} - e^{2\frac{\nu\ln(H/S)}{\sigma^2}}\frac{1}{\sigma\sqrt{2\pi T}}\, e^{-\frac{1}{2}\left(\frac{z-\ln S-2\ln(H/S)-\nu T}{\sigma\sqrt{T}}\right)^2}\right] \tag{2}$$

where H is the barrier level.

By the time-homogeneity of the exchange rate dynamics, we have that

$$Q^d\Big\{X_T \in dx, \max_{u\in[t,T]}(\omega X_u) \leq \omega y|\mathcal{F}_t\Big\} = Q^d\big\{X_\tau \in dx, M_\tau^X \leq \omega y\big\}$$

Therefore, the barrier option price at time t, given that the barrier has not been hit before, is

$$V(t, S_t, r, q, \sigma, \tau) = e^{-r\tau}\int_{z\in A} h(e^z) f(z; S_t, r, q, \sigma, \tau)\, dz \tag{3}$$

which depends on t only through τ.

1.3 The Black and Scholes P.D.E.

The derivative's price is assumed to be a smooth function of time and exchange rate. From the standard Black and Scholes theory,[1] the function V then follows

$$\boxed{\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV} \tag{4}$$

with terminal condition $V_T = h(S_T)$.

1.4 The Time Homogeneity

From (3), we immediately see that the derivative's price depends on t only implicitly through the time-to-maturity $\tau = T - t$. This leads to

$$\boxed{\frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau}} \tag{5}$$

1.5 The Scale Invariance of Time

From (3) we also see that replacing r, q, σ and τ as follows

$$\begin{aligned} r &\to r\alpha \\ q &\to q\alpha \\ \sigma &\to \sigma\sqrt{\alpha} \\ \tau &\to \frac{\tau}{\alpha} \end{aligned} \tag{6}$$

where α is any (strictly) positive real number, the derivative price does not change, namely

$$V(t, S_t, r\alpha, q\alpha, \sigma\sqrt{\alpha}, \tau/\alpha) = V(t, S_t, r, q, \sigma, \tau)$$

Since this relation holds for any α, we can differentiate both sides with respect to α and then set $\alpha = 1$. We get

$$\frac{1}{2}\sigma\frac{\partial V}{\partial \sigma} + r\frac{\partial V}{\partial r} + q\frac{\partial V}{\partial q} - \tau\frac{\partial V}{\partial \tau} = 0$$

or equivalently, by (5),

$$\boxed{\frac{1}{2}\sigma\frac{\partial V}{\partial \sigma} + r\frac{\partial V}{\partial r} + q\frac{\partial V}{\partial q} + \tau\frac{\partial V}{\partial t} = 0} \tag{7}$$

This relationship has been derived by Reiss and Wystup (2001).

[1] Equivalently, by (1) and the Feynman-Kac formula.

1.6 The Interest Rates Symmetry

Differentiating (3) with respect to r and q, we get

$$\frac{\partial V}{\partial r} = -\tau V + e^{-r\tau}\int_{z\in A} h(e^z)\frac{\partial}{\partial r}f(z; S_t, r, q, \sigma, \tau)\, dz$$
$$\frac{\partial V}{\partial q} = e^{-r\tau}\int_{z\in A} h(e^z)\frac{\partial}{\partial q}f(z; S_t, r, q, \sigma, \tau)\, dz$$

Noting that

$$\frac{\partial}{\partial q}f(z; S_t, r, q, \sigma, \tau) = -\frac{\partial}{\partial r}f(z; S_t, r, q, \sigma, \tau)$$

the sensitivities of the derivative's price with respect to the two instantaneous rates then satisfy

$$\boxed{\frac{\partial V}{\partial r} + \frac{\partial V}{\partial q} + \tau V = 0} \tag{8}$$

1.7 The Vega-Gamma Relationship

From (4) we obtain

$$\frac{\partial V}{\partial t} = rV - (r-q)S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}$$

which substituted into (7) yields

$$\frac{1}{2}\sigma\frac{\partial V}{\partial \sigma} + r\frac{\partial V}{\partial r} + q\frac{\partial V}{\partial q} + \tau\Big[rV - (r-q)S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\Big] = 0$$

Rearranging terms we get

$$\frac{1}{2}\sigma\frac{\partial V}{\partial \sigma} = \frac{\tau}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + (r-q)\tau S\frac{\partial V}{\partial S} - r\Big(\frac{\partial V}{\partial r} + \tau V\Big) - q\frac{\partial V}{\partial q}$$

Applying (8) and collecting terms, we finally obtain

$$\boxed{\frac{1}{2}\sigma\frac{\partial V}{\partial \sigma} = \frac{\tau}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + (r-q)\Big[\tau S\frac{\partial V}{\partial S} + \frac{\partial V}{\partial q}\Big]} \tag{9}$$

Remark 1.1. *The Vega-Gamma relationship (9) can also be proved by noting that an analogous relationship also holds for the density* $f(z; S, r, q, \sigma, \tau)$, i.e.

$$\frac{1}{2}\sigma\frac{\partial f}{\partial \sigma} = \frac{\tau}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} + (r-q)\Big[\tau S\frac{\partial f}{\partial S} + \frac{\partial f}{\partial q}\Big]$$

This simply follows by the calculation of the related derivatives.

Remark 1.2. *In the case of a European-style payoff, using a property of the lognormal distribution, we can also prove that*

$$\frac{\partial V}{\partial q} = -\tau S \frac{\partial V}{\partial S}$$

so that the Vega-Gamma relationship for European-style options reads as

$$\frac{1}{2}\sigma \frac{\partial V}{\partial \sigma} = \frac{\tau}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

References

[1] Black, F. and Scholes, M. (1973) The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* 81, 637-659.

[2] Musiela, M. and Rutkowski, M. (1998) Martingale Methods in Financial Modelling. Springer. Berlin.

[3] Reiss, O. and Wystup, U. (2001) Computing Option Price Sensitivities Using Homogeneity and Other Tricks, *The Journal of Derivatives* 9(2), 41 - 53.