

# A Vega-Gamma Relationship for European-Style or Barrier Options in the Black-Scholes Model

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## Abstract

In this document we derive some fundamental relationships between the Greeks of a barrier option under the Black-Scholes model. A European-style option can be considered as a limit case. Besides the classical Black and Scholes P.D.E., a barrier option price is shown to also satisfy: i) a time homogeneity; ii) a scale invariance of time; iii) an interest rates symmetry; iv) a Vega-Gamma relationship.

## Notation

$t$ : the running time.

$S_t$ : the price at time  $t$  in domestic currency of one unit of foreign currency.

$r$ : the (constant) domestic instantaneous risk-free rate.

$q$ : the (constant) foreign instantaneous risk-free rate.

$\sigma$ : the exchange rate (constant) percentage volatility.

$T$ : the maturity of a given derivative.

$\tau$ : the time-to-maturity of this given derivative, namely  $\tau = T - t$ .

$V_T$ : the derivative's payoff at time  $T$ .

$V(t, S_t, r, q, \sigma, \tau)$ : no-arbitrage price at time  $t$  of a given derivative security with time-to-maturity  $\tau$ , when the underlying exchange rate is  $S$ , the domestic instantaneous risk-free rate is  $r$ , the foreign instantaneous risk-free rate is  $q$  and the exchange rate percentage volatility is  $\sigma$ . A shorthand notation for it is  $V_t$ .

$Q^d$ : the domestic risk-neutral measure.

$E^d$ : expectation under  $Q^d$ .

$\mathcal{F}_t$ : the  $\sigma$ -algebra generated by  $S$  up to time  $t$ .

## 1.1 Assumptions

The exchange rate  $S$  is assumed to evolve under the domestic risk-neutral measure  $Q^d$  according to:

$$dS_t = S_t[(r - q) dt + \sigma dW_t]$$

where  $W$  is a standard Brownian motion under  $Q^d$ .

We are given a derivative security with payoff  $V_T$  at time  $T$ . This derivative is either a European-style option or a (single) barrier option. We actually restrict our analysis to

“out” barrier options, which become worthless if the barrier is hit prior to maturity. In fact, a European-style payoff can be obtained by letting the barrier level either go to zero or to infinity, whereas an “in” barrier option is simply the difference between the corresponding European-style and “out” barrier options.

## 1.2 The pricing of a Barrier Option

Standard theory in derivatives pricing implies that the no-arbitrage price at time  $t$  of the payoff  $V_T$  at time  $T$  is

$$V_t = e^{-r\tau} E^d[V_T | \mathcal{F}_t] \quad (1)$$

Set  $\nu := r - q - \frac{1}{2}\sigma^2$  and

$$\begin{aligned} X_t &:= \nu t + \sigma W_t \\ M_t^X &:= \max_{u \in [0, t]} (\omega X_u) \end{aligned}$$

where  $\omega = 1$  for maximum and  $\omega = -1$  for minimum. Then, see for instance Musiela and Rutkowski (1998), for  $\omega y \geq \max(0, \omega x)$ ,

$$Q^d \{X_T \in dx, M_T^X \leq \omega y\} = \omega \left[ \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{1}{2}\left(\frac{x-\nu T}{\sigma\sqrt{T}}\right)^2} - e^{2\frac{\nu y}{\sigma^2}} \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{1}{2}\left(\frac{x-2y-\nu T}{\sigma\sqrt{T}}\right)^2} \right]$$

Assuming that the option payoff, if the barrier is not hit prior to maturity, is a function  $h(S_T)$  of the underlying at maturity, the barrier option price at time 0 can be written as

$$V_0 = e^{-rT} \int_{z \in A} h(e^z) f(z; S_0, r, q, \sigma, T) dz$$

where neither  $h$  nor  $A \subset \mathbb{R}$  depend on  $S_0, r, q, \sigma, T$  and

$$f(z; S, r, q, \sigma, T) = \omega \left[ \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{1}{2}\left(\frac{z-\ln S-\nu T}{\sigma\sqrt{T}}\right)^2} - e^{2\frac{\nu \ln(H/S)}{\sigma^2}} \frac{1}{\sigma\sqrt{2\pi T}} e^{-\frac{1}{2}\left(\frac{z-\ln S-2\ln(H/S)-\nu T}{\sigma\sqrt{T}}\right)^2} \right] \quad (2)$$

where  $H$  is the barrier level.

By the time-homogeneity of the exchange rate dynamics, we have that

$$Q^d \left\{ X_T \in dx, \max_{u \in [t, T]} (\omega X_u) \leq \omega y | \mathcal{F}_t \right\} = Q^d \{ X_\tau \in dx, M_\tau^X \leq \omega y \}$$

Therefore, the barrier option price at time  $t$ , given that the barrier has not been hit before, is

$$V(t, S_t, r, q, \sigma, \tau) = e^{-r\tau} \int_{z \in A} h(e^z) f(z; S_t, r, q, \sigma, \tau) dz \quad (3)$$

which depends on  $t$  only through  $\tau$ .

### 1.3 The Black and Scholes P.D.E.

The derivative's price is assumed to be a smooth function of time and exchange rate. From the standard Black and Scholes theory,<sup>1</sup> the function  $V$  then follows

$$\boxed{\frac{\partial V}{\partial t} + (r - q)S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV} \quad (4)$$

with terminal condition  $V_T = h(S_T)$ .

### 1.4 The Time Homogeneity

From (3), we immediately see that the derivative's price depends on  $t$  only implicitly through the time-to-maturity  $\tau = T - t$ . This leads to

$$\boxed{\frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau}} \quad (5)$$

### 1.5 The Scale Invariance of Time

From (3) we also see that replacing  $r$ ,  $q$ ,  $\sigma$  and  $\tau$  as follows

$$\begin{aligned} r &\rightarrow r\alpha \\ q &\rightarrow q\alpha \\ \sigma &\rightarrow \sigma\sqrt{\alpha} \\ \tau &\rightarrow \frac{\tau}{\alpha} \end{aligned} \quad (6)$$

where  $\alpha$  is any (strictly) positive real number, the derivative price does not change, namely

$$V(t, S_t, r\alpha, q\alpha, \sigma\sqrt{\alpha}, \tau/\alpha) = V(t, S_t, r, q, \sigma, \tau)$$

Since this relation holds for any  $\alpha$ , we can differentiate both sides with respect to  $\alpha$  and then set  $\alpha = 1$ . We get

$$\frac{1}{2}\sigma \frac{\partial V}{\partial \sigma} + r \frac{\partial V}{\partial r} + q \frac{\partial V}{\partial q} - \tau \frac{\partial V}{\partial \tau} = 0$$

or equivalently, by (5),

$$\boxed{\frac{1}{2}\sigma \frac{\partial V}{\partial \sigma} + r \frac{\partial V}{\partial r} + q \frac{\partial V}{\partial q} + \tau \frac{\partial V}{\partial t} = 0} \quad (7)$$

This relationship has been derived by Reiss and Wystup (2001).

<sup>1</sup>Equivalently, by (1) and the Feynman-Kac formula.

## 1.6 The Interest Rates Symmetry

Differentiating (3) with respect to  $r$  and  $q$ , we get

$$\begin{aligned}\frac{\partial V}{\partial r} &= -\tau V + e^{-r\tau} \int_{z \in A} h(e^z) \frac{\partial}{\partial r} f(z; S_t, r, q, \sigma, \tau) dz \\ \frac{\partial V}{\partial q} &= e^{-r\tau} \int_{z \in A} h(e^z) \frac{\partial}{\partial q} f(z; S_t, r, q, \sigma, \tau) dz\end{aligned}$$

Noting that

$$\frac{\partial}{\partial q} f(z; S_t, r, q, \sigma, \tau) = -\frac{\partial}{\partial r} f(z; S_t, r, q, \sigma, \tau)$$

the sensitivities of the derivative's price with respect to the two instantaneous rates then satisfy

$$\boxed{\frac{\partial V}{\partial r} + \frac{\partial V}{\partial q} + \tau V = 0} \quad (8)$$

## 1.7 The Vega-Gamma Relationship

From (4) we obtain

$$\frac{\partial V}{\partial t} = rV - (r - q)S \frac{\partial V}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

which substituted into (7) yields

$$\frac{1}{2} \sigma \frac{\partial V}{\partial \sigma} + r \frac{\partial V}{\partial r} + q \frac{\partial V}{\partial q} + \tau \left[ rV - (r - q)S \frac{\partial V}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right] = 0$$

Rearranging terms we get

$$\frac{1}{2} \sigma \frac{\partial V}{\partial \sigma} = \frac{\tau}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) \tau S \frac{\partial V}{\partial S} - r \left( \frac{\partial V}{\partial r} + \tau V \right) - q \frac{\partial V}{\partial q}$$

Applying (8) and collecting terms, we finally obtain

$$\boxed{\frac{1}{2} \sigma \frac{\partial V}{\partial \sigma} = \frac{\tau}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) \left[ \tau S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial q} \right]} \quad (9)$$

**Remark 1.1.** *The Vega-Gamma relationship (9) can also be proved by noting that an analogous relationship also holds for the density  $f(z; S, r, q, \sigma, \tau)$ , i.e.*

$$\frac{1}{2} \sigma \frac{\partial f}{\partial \sigma} = \frac{\tau}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (r - q) \left[ \tau S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial q} \right]$$

*This simply follows by the calculation of the related derivatives.*

**Remark 1.2.** *In the case of a European-style payoff, using a property of the lognormal distribution, we can also prove that*

$$\frac{\partial V}{\partial q} = -\tau S \frac{\partial V}{\partial S}$$

*so that the Vega-Gamma relationship for European-style options reads as*

$$\frac{1}{2}\sigma \frac{\partial V}{\partial \sigma} = \frac{\tau}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

## References

- [1] Black, F. and Scholes, M. (1973) The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* 81, 637-659.
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- [3] Reiss, O. and Wystup, U. (2001) Computing Option Price Sensitivities Using Homogeneity and Other Tricks, *The Journal of Derivatives* 9(2), 41 - 53.