A multi-stage uncertain-volatility model

Fabio Mercurio
Financial Models
Banca IMI
Corso Matteotti, 6
20121 Milano, Italy

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Abstract

We consider a simple uncertain-volatility model for the asset price underlying a given option market. The asset price volatility is assumed to follow a discrete (actually finite) Markov chain σ, which changes value on some fixed future times. The volatility chain is independent of the Brownian motion governing the future evolution of the asset. Modeling the volatility evolution in this way is equivalent to assuming different possible scenarios for the asset forward volatility.

1 Introduction

The success of the Black-Scholes formula is mainly due to the possibility of synthesizing option prices through a unique parameter, the implied volatility, which is so crucial for traders to be directly quoted itself in many financial markets. This is because the Black-Scholes formula allows one to immediately convert a volatility into the price at which the related option can be exchanged.

The Black-Scholes model, however, can not consistently price all options traded in a specific market. In fact, the assumption of a deterministic underlying-asset volatility leads to constant implied volatilities for any fixed maturity, in contrast with the smile/skew effect commonly observed in practice. Moreover, historical analysis shows that volatilities are indeed stochastic (often correlated with the underlying-asset price).

Stochastic volatility models, therefore, seem to represent a more realistic choice when modeling asset price dynamics for valuing derivative securities. However, only few examples, see Hull and White (1987) or Heston (1993), retain enough analytical tractability. In general, in fact, the calibration to market option prices, and the consequent book re-evaluation, can be extremely burdensome and time consuming.

Stochastic volatility models can also be problematic as far as hedging is concerned: hedging volatility changes is less straightforward than in the Black-Scholes case where we just have one volatility parameter. In general, we can only calculate the sensitivity with respect to the model parameters, which may have an economic meaning but are likely not to have a clear impact on an implied volatility surface.

The purpose of this paper is to propose a stochastic-volatility model that is analytically tractable as much as Black and Scholes's and for which Vega hedging can be defined in a natural way. The model is based on an uncertain volatility whose random value is drawn, on some fixed future times, from a finite distribution. Such a value is known and constant between a draw and the next. The model is similar in spirit to that of Alexander and Brintalos (2003).

Our uncertain volatility model is equivalent to assuming a number of possible different scenarios for the asset forward volatility, which can therefore be hedged accordingly. As a direct consequence, European-style claims are simply mixtures of the corresponding prices in the Black-Scholes model.

The paper is structured as follows. Section 2 introduces the model. Section 3 proves the absence of arbitrage and calculates prices of derivative securities.

2 The model description

An asset price is assumed to evolve (under the real world) according to

$$dS(t) = S(t)[\mu(t)\,dt + \sigma(t)\,dW(t)], \quad S(0) = S_0 > 0, \tag{1}$$

where W is a standard Brownian motion on a probability space $(\Omega^W, \mathcal{F}^W, P^W)$, μ is a deterministic function of time and σ is the finite-space stochastic process defined as follows. Given a finite time horizon T and a set of M fixed times $\mathcal{T} = \{T_1, T_2, \ldots, T_M\}$, with $T_1 > 0$ and $T_M < T$, we assume that $\sigma(t) = \sigma_0 > 0$ for $t \in [0, T_1)$ and $\sigma(t) = \sigma_k$ for $t \in [T_k, T_{k+1})$, $k = 1, \ldots, M-1$, where σ_k is the following discrete random variable

$$\sigma_k = \begin{cases} \varsigma_1^k & p_1^k \\ \varsigma_2^k & p_2^k \\ \vdots & \vdots \\ \varsigma_N^k & p_N^k \end{cases} \tag{2}$$

which is defined on a probability space $(\Omega_k^\sigma, \mathcal{F}_k^\sigma, P_k^\sigma)$.[1] More synthetically, we write $\sigma(t) = \sigma_{\kappa(t)}$, with $\kappa(t) = k$ for each $t \in [T_k, T_{k+1})$ and $T_0 := 0$. The values ς_i^k's and the probabilities p_i^k's are strictly positive with $\sum_{i=1}^N p_i^k = 1$ for each k. Each random variable σ_k is assumed to be independent of the others and of W.

We denote by $\omega_{k,i}^\sigma$ the i-th point in Ω_k^σ, so that $\sigma_k(\omega_{k,i}^\sigma) = \varsigma_i^k$ and $P_k^\sigma(\{\omega_{k,i}^\sigma\}) = p_i^k$, $i = 1, \ldots, N$, $k = 1, \ldots, M$. We set $\Omega^\sigma := \Omega_1^\sigma \times \cdots \times \Omega_M^\sigma$, $P^\sigma := P_1^\sigma \otimes \cdots \otimes P_M^\sigma$ and

[1] We can also take different N's for different k's or assume that the ς's are deterministic function of time.

$\mathcal{F}^\sigma := \mathcal{F}_1^\sigma \otimes \cdots \otimes \mathcal{F}_M^\sigma$.[2] We thus have $P^\sigma(\{(\omega^\sigma_{1,i_1}, \omega^\sigma_{2,i_2}, \ldots, \omega^\sigma_{M,i_M})\}) = p^1_{i_1} p^2_{i_2} \cdots p^M_{i_M}$ for each $(\omega^\sigma_{1,i_1}, \omega^\sigma_{2,i_2}, \ldots, \omega^\sigma_{M,i_M}) \in \Omega^\sigma$. We denote by $\mathcal{F}_t^W$ and $\mathcal{F}_t^\sigma$ the σ-fields generated, respectively, by W and σ up to time t.

Since W and σ are independent we take as underlying filtration $\mathbb{F} := \{\mathcal{F}_t = \mathcal{F}_t^W \otimes \mathcal{F}_t^\sigma : 0 \le t \le T\}$. We set $\Omega := \Omega^W \times \Omega^\sigma$ and $P := P^W \otimes P^\sigma$. The asset price (1) is then defined on the (real-world) probability space $(\Omega, \mathcal{F}_T, P)$.

3 Absence of arbitrage and derivatives pricing

We denote by $r(t)$ the deterministic instantaneous short rate at time t, and by B the bank-account numeraire, $B(t) = \exp\{\int_0^t r(u)\, du\}$. The asset S is assumed to pay a deterministic dividend yield $y(t)$ at time t. For instance, in case S represents an exchange rate, $y(t) = r_f(t)$, where r_f denotes the deterministic instantaneous short rate for the foreign currency.

Proposition 3.1. *Let us define a new measure on $(\Omega^\sigma, \mathcal{F}^\sigma)$ by*

$$\frac{dQ^\sigma}{dP^\sigma}\big((\omega^\sigma_{1,i_1}, \omega^\sigma_{2,i_2}, \ldots, \omega^\sigma_{M,i_M})\big) = \frac{\lambda^1_{i_1}\lambda^2_{i_2}\cdots\lambda^M_{i_M}}{p^1_{i_1}p^2_{i_2}\cdots p^M_{i_M}}, \quad (i_1, i_2, \ldots, i_M) \in \{1, 2, \ldots, N\}^M, \tag{3}$$

where the new probabilities λ's are strictly positive and $\sum_{i=1}^N \lambda_i^k = 1$ for each k. Then there exists a risk-neutral measure Q on $(\Omega, \mathcal{F}_T)$ associated with the numeraire B_y, $B_y(t) = \exp\{\int_0^t (r(u) - y(u))\, du\}$,

$$\frac{dQ}{dP} = \frac{dQ^\sigma}{dP^\sigma} \exp\left\{-\frac{1}{2}\int_0^T \left(\frac{\mu(t) - r(t) + y(t)}{\sigma(t)}\right)^2 dt - \int_0^T \frac{\mu(t) - r(t) + y(t)}{\sigma(t)}\, dW(t)\right\}, \tag{4}$$

so that the asset has, under Q, a drift rate equal to $r(t) - y(t)$.

Proof. The intuition behind definition (4) is as follows. Under each volatility scenario, given by a point in Ω^σ, S evolves according to a geometric Brownian motion with a piecewise-constant volatility. It is then enough to apply Girsanov theorem combined with a measure change for a finite-space process. More formally, the process S/B_y is an $(\mathcal{F}_T, Q)$-martingale

[2]Let $(E, \mathcal{E}, P)$ and $(F, \mathcal{F}, Q)$ be two probability spaces. Then $\mathcal{E} \otimes \mathcal{F}$ denotes the σ-algebra generated by $\mathcal{E} \times \mathcal{F}$, with $\times$ denoting Cartesian product, while $P \otimes Q$ denotes the unique extension on $\mathcal{E} \otimes \mathcal{F}$ of the probability R such that $R(A \times B) = p(A)Q(B)$ for $A \in \mathcal{E}$ and $B \in \mathcal{F}$. These definitions extend naturally to the cases with more than two spaces.

since, for each $0 \le t < T$,

$$\frac{E\left[S(T)\, e^{\int_0^T (y(u)-r(u))\,du} \frac{dQ}{dP}\Big|\mathcal{F}_t\right]}{E\left[\frac{dQ}{dP}\Big|\mathcal{F}_t\right]} = \frac{E\left\{E\left[e^{\int_0^T (y(u)-r(u))\,du} S(T)\, \frac{dQ}{dP}\Big|\mathcal{F}_t^W \otimes \mathcal{F}^\sigma\right]\Big|\mathcal{F}_t\right\}}{E\left\{E\left[\frac{dQ}{dP}\Big|\mathcal{F}_t^W \otimes \mathcal{F}^\sigma\right]\Big|\mathcal{F}_t\right\}}$$

$$= \frac{E\left\{\frac{dQ^\sigma}{dP^\sigma}\, E\left[S(T)\, e^{\int_0^T (y(u)-r(u))\,du - \frac{1}{2}\int_0^T\left(\frac{\mu(u)-r(u)+y(u)}{\sigma(u)}\right)^2 du - \int_0^T \frac{\mu(u)-r(u)+y(u)}{\sigma(u)}\,dW(u)}\Big|\mathcal{F}_t^W \otimes \mathcal{F}^\sigma\right]\Big|\mathcal{F}_t\right\}}{E\left\{\frac{dQ^\sigma}{dP^\sigma}\, e^{-\frac{1}{2}\int_0^t\left(\frac{\mu(u)-r(u)+y(u)}{\sigma(u)}\right)^2 du - \int_0^t \frac{\mu(u)-r(u)+y(u)}{\sigma(u)}\,dW(u)}\Big|\mathcal{F}_t\right\}}$$

$$= \frac{E\left\{\frac{dQ^\sigma}{dP^\sigma}\, E\left[S(T)\, e^{\int_0^T (y(u)-r(u))\,du - \frac{1}{2}\int_t^T\left(\frac{\mu(u)-r(u)+y(u)}{\sigma(u)}\right)^2 du - \int_t^T \frac{\mu(u)-r(u)+y(u)}{\sigma(u)}\,dW(u)}\Big|\mathcal{F}_t^W \otimes \mathcal{F}^\sigma\right]\Big|\mathcal{F}_t\right\}}{E\left\{\frac{dQ^\sigma}{dP^\sigma}\Big|\mathcal{F}_t\right\}}$$

$$= \frac{E\left\{\frac{dQ^\sigma}{dP^\sigma}\, S(t)\, e^{\int_0^t (y(u)-r(u))\,du}\Big|\mathcal{F}_t\right\}}{E\left\{\frac{dQ^\sigma}{dP^\sigma}\Big|\mathcal{F}_t\right\}}$$

$$= S(t)\, e^{\int_0^t (y(u)-r(u))\,du}.$$

□

Thanks to the fundamental results of Harrison and Kreps (1979) and Harrison and Pliska (1981), the existence of an equivalent martingale measure ensures the absence of arbitrage opportunities. Once we have selected the risk-neutral probabilities λ's, we can then price contingent claims according to the following.

Proposition 3.2. *Every claim written on S with an $\mathcal{F}_T$-measurable (and square integrable) payoff H at time T has a no-arbitrage price at time t given by*

$$H_t = e^{-R(t,T)} \sum_{i_{\kappa(t)+1},\dots,i_M=1}^{N} \lambda_{i_{\kappa(t)+1}}^{\kappa(t)+1} \cdots \lambda_{i_M}^{M}\, E^Q\left\{H\Big|\mathcal{F}_t, \{\sigma_{\kappa(t)+1} = \varsigma_{i_{\kappa(t)+1}}^{\kappa(t)+1}\}, \dots, \{\sigma_M = \varsigma_{i_M}^{M}\}\right\}, \tag{5}$$

where $R(t,T) := \int_t^T r(u)\,du$ and E^Q denotes expectation under the measure Q in (4).

Proof. Under the risk-neutral measure Q, the no-arbitrage claim price is

$$H_t = e^{-R(t,T)} E^Q\{H|\mathcal{F}_t\}.$$

Conditioning on the whole path of σ, we then have

$$H_t = e^{-R(t,T)} E^Q\left\{E^Q\left[H|\mathcal{F}_t^W \otimes \mathcal{F}^\sigma\right]|\mathcal{F}_t\right\},$$

from which (5) follows. □

Corollary 3.3. *Consider a European option with maturity T, strike K and written on the asset. The option value at time t, when the asset price is S_t is then given by the following convex combination of Black-Scholes prices*

$$\begin{aligned}\mathrm{EO}(t;S_t,K,T) = \omega & \sum_{i_{\kappa(t)+1},\dots,i_M=1}^{N} \lambda^{\kappa(t)+1}_{i_{\kappa(t)+1}} \cdots \lambda^{M}_{i_M} \\ & \cdot \Bigg[S_0 e^{-Y(t,T)} \Phi\left(\omega \frac{\ln\frac{S_0}{K} + R(t,T) - Y(t,T) + \frac{1}{2} V^2_{i_{\kappa(t)+1},\cdots,i_M}(t,T)}{V_{i_{\kappa(t)+1},\cdots,i_M}(t,T)} \right) \\ & \quad - K e^{-R(t,T)} \Phi\left(\omega \frac{\ln\frac{S_0}{K} + R(t,T) - Y(t,T) - \frac{1}{2} V^2_{i_{\kappa(t)+1},\cdots,i_M}(t,T)}{V_{i_{\kappa(t)+1},\cdots,i_M}(t,T)} \right) \Bigg],\end{aligned} \tag{6}$$

with Φ denoting the standard normal cumulative distribution function, and where $\omega = 1$ for a call and $\omega = -1$ for a put, $Y(t,T) := \int_t^T y(u)\,du$ and

$$V_{i_{\kappa(t)+1},\cdots,i_M}(t,T) := \sqrt{(\sigma_{\kappa(t)})^2 (T_{\kappa(t)+1} - t) + \sum_{k=\kappa(t)+1}^{M-1} (\varsigma^k_{i_k})^2 (T_{k+1} - T_k) + (\varsigma^M_{i_M})^2 (T - T_M)}. \tag{7}$$

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