Pricing of Options on two Currencies Libor Rates

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Abstract

In this document we show how to price options on two Libor rates belonging to two different currencies (the former is domestic, the latter foreign). To this end, we explicitly derive the dynamics of the foreign rate under the domestic forward measure associated to the rate maturity.

We then consider the fundamental case of an option written on the spread between the two Libor rates and derive closed form formulas for both the "up-front" and the "in-arrears" cases. Explicit formulas are also derived for options on the product of the two rates as well as for trigger swaps.

1.1 Assumptions and Definitions

Given a domestic market and a foreign market, let us assume that the term structures of discount factors that are observed in the domestic and foreign markets at time t are respectively given by $T \mapsto P(t,T)$ and $T \mapsto P^f(t,T)$ for $T \geq t$. Let us denote by $\mathcal{X}(t)$ the exchange rate at time t between the currencies in the two markets, in that 1 unit of the foreign currency equals $\mathcal{X}(t)$ units of the domestic currency.

Given the future times T_{i-1} and T_i, $i = 1, \ldots, n$, the domestic and foreign forward rates at time t for the interval $[T_{i-1}, T_i]$ are, respectively,

$$F_i(t) = F(t; T_{i-1}, T_i) = \frac{P(t, T_{i-1}) - P(t, T_i)}{\tau_i P(t, T_i)}$$
$$F_i^f(t) = F^f(t; T_{i-1}, T_i) = \frac{P^f(t, T_{i-1}) - P^f(t, T_i)}{\tau_i P^f(t, T_i)}$$

where τ_i is the year fraction between times T_{i-1} and T_i, which is assumed to be the same in both markets.

Denoting by $F_{\mathcal{X}}(t, T_i)$ the forward exchange rate at time t for maturity T_i,

$$F_{\mathcal{X}}(t, T_i) = \mathcal{X}(t) \frac{P^f(t, T_i)}{P(t, T_i)},$$

and assuming constant (proportional) volatilities, the two forward rates evolve under the domestic forward measure Q^i according to (see Brigo and Mercurio, 2001, Sections 6.3 and 11.4)

$$dF_i(t) = \sigma_i F_i(t)\, dW_i(t),$$
$$dF_i^f(t) = F_i^f(t)\big[-\rho\sigma_{F\mathcal{X}}\sigma_i^f\, dt + \sigma_i^f\, dW_i^f(t)\big],$$

where W_i and W_i^f are two standard Brownian motions with instantaneous correlation ρ_i, ρ is the instantaneous correlation between $F_{\mathcal{X}}(\cdot, T_i)$ and $F_i^f(\cdot)$, and $\sigma_{F\mathcal{X}}$ is the assumed constant (proportional) volatility of the forward exchange rate $F_{\mathcal{X}}(t, T_i)$:

$$dF_{\mathcal{X}}(t, T_i) = \sigma_{F\mathcal{X}} F_{\mathcal{X}}(t, T_i)\, dW_{\mathcal{X}}(t),$$

where $W_{\mathcal{X}}$ is a standard Brownian motion under Q^i, with $dW_{\mathcal{X}}(t)dW_i^f(t) = \rho\, dt$.[1]

Let us consider a derivative whose payoff at time T_i is a function $g(F_i(T_{i-1}), F_i^f(T_{i-1}))$. By formula (2.21) in Brigo and Mercurio (2001), the no-arbitrage value at time t of such a payoff is

$$P(t, T_i)E^i\big\{g(F_i(T_{i-1}), F_i^f(T_{i-1}))|\mathcal{F}_t\big\}, \tag{1}$$

where E^i denotes expectation under Q^i and $\mathcal{F}_t$ is the σ-field generated by the pair (F_i, F_i^f) up to time t.

1.2 Spread Options

A spread option on the two Libor rates $L(T_{i-1}, T_i)$ and $L^f(T_{i-1}, T_i)$ is a derivative paying off at time T_i, in domestic currency,

$$\tau_i N\left[\omega\big(L(T_{i-1}, T_i) - L^f(T_{i-1}, T_i) + K\big)\right]^+ = \tau_i N\left[\omega\Big(F_i(T_{i-1}) - F_i^f(T_{i-1}) + K\Big)\right]^+, \tag{2}$$

where N is the nominal value, K is the contract margin and $\omega = 1$ for a call and $\omega = -1$ for a put.

An "in-arrears" spread option pays off the same quantity at time T_{i-1}. This is equivalent to paying off at time T_i

$$\tau_i N\left[\omega\Big(F_i(T_{i-1}) - F_i^f(T_{i-1}) + K\Big)\right]^+ \big(1 + \tau_i F_i(T_{i-1})\big). \tag{3}$$

The two payoffs (2) and (3) can be summarized into

$$\tau_i N\left[\omega\Big(F_i(T_{i-1}) - F_i^f(T_{i-1}) + K\Big)\right]^+ \big(1 + \psi\tau_i F_i(T_{i-1})\big), \tag{4}$$

where $\psi = 1$ for the "in-arrears" case and $\psi = 0$ otherwise.

[1] Notice that $F_{\mathcal{X}}$ is a martingale under Q^i.

Proposition 1.1. *The no-arbitrage value at time t of the payoff (4) is given by*

$$\mathbf{LSO}(t, T_{i-1}, T_i, \tau_i, N, K, \omega, \psi) = \tau_i N P(t, T_i) \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} \big[1 + \psi\tau_i\big(h(v) - K\big)\big] f(v)\, dv, \tag{5}$$

where

$$\begin{aligned} f(v) = \Bigg[&-\omega F_i^f(t) e^{\mu_y + \rho_i\sigma_y v + \frac{1}{2}\sigma_y^2(1-\rho_i^2)} \Phi\left(-\omega \frac{\ln \frac{F_i^f(t)}{h(v)} + \mu_y + \rho_i\sigma_y v + \sigma_y^2(1-\rho_i^2)}{\sigma_y\sqrt{1-\rho_i^2}}\right) \\ &+\omega h(v)\Phi\left(-\omega \frac{\ln \frac{F_i^f(t)}{h(v)} + \mu_y + \rho_i\sigma_y v}{\sigma_y\sqrt{1-\rho_i^2}}\right)\Bigg] 1_{\{h(v)>0\}} \\ &+ \tfrac{1}{2}(1-\omega)1_{\{h(v)\le 0\}} \left[-h(v) + F_i^f(t) e^{\mu_y + \rho_i\sigma_y v + \frac{1}{2}\sigma_y^2(1-\rho_i)^2}\right] \end{aligned}$$

with 1_A denoting the indicator function of the set A, $\Phi(\cdot)$ denoting the standard normal cumulative distribution function, and

$$\begin{aligned} h(v) &= K + F_i(t) e^{\mu_x + \sigma_x v} \\ \mu_x &= -\tfrac{1}{2}\sigma_x^2 \\ \mu_y &= -\rho\sigma_{F\mathcal{X}}\sigma_i^f\tau - \tfrac{1}{2}\sigma_y^2 \\ \sigma_x &= \sigma_i\sqrt{\tau} \\ \sigma_y &= \sigma_i^f\sqrt{\tau} \\ \tau &= T_{i-1} - t \end{aligned}$$

Proof. By formula (1), the no-arbitrage value at time t of the payoff (4) is

$$\tau_i N P(t, T_i) E^i \left\{ \Big[\omega\big(F_i(T_{i-1}) - F_i^f(T_{i-1}) + K\big)\Big]^+ \big(1 + \psi\tau_i F_i(T_{i-1})\big) | \mathcal{F}_t \right\}. \tag{6}$$

Defining

$$X := \ln \frac{F_i(T_{i-1})}{F_i(t)}, \qquad Y := \ln \frac{F_i^f(T_{i-1})}{F_i^f(t)},$$

the joint density function $f_{X,Y}$ of (X, Y) under the measure Q^i is bivariate normal with mean vector and variance-covariance matrix respectively given by

$$M_{X,Y} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \quad V_{X,Y} = \begin{bmatrix} \sigma_x^2 & \rho_i\sigma_x\sigma_y \\ \rho_i\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$$

that is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_i^2}} \exp\left[-\frac{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho_i \frac{x-\mu_x}{\sigma_x}\frac{y-\mu_y}{\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2}{2(1-\rho_i^2)}\right].$$

It is well known that

$$f_{X,Y}(x,y) = f_{Y|X}(x,y) f_X(x),$$

where

$$\begin{aligned} f_{Y|X}(x,y) &= \frac{1}{\sigma_y\sqrt{2\pi}\sqrt{1-\rho_i^2}} \exp\left[-\frac{\left(\frac{y-\mu_y}{\sigma_y} - \rho_i \frac{x-\mu_x}{\sigma_x}\right)^2}{2(1-\rho_i^2)}\right] \\ f_X(x) &= \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right]. \end{aligned} \tag{7}$$

The expectation in (6) can thus be written as

$$\int_{-\infty}^{+\infty} \left(1+\psi\tau_i F_i(t)e^x\right)\left[\int_{-\infty}^{+\infty} \left(\omega F_i(t)e^x - \omega F_i^f(t)e^y + \omega K\right)^+ f_{Y|X}(x,y)\,dy\right] f_X(x)\,dx$$

The expression between square brackets can be calculated analytically by distinguishing two cases:

1. $F_i(t)e^x + K \leq 0$.

   If $\omega = 1$, the expression is equal to 0 (the positive part of a negative number is zero). If $\omega = -1$, instead,

$$\begin{aligned} \left[\cdots\right] &= -F_i(t)e^x - K + F_i^f(t)\int_{-\infty}^{+\infty} e^y f_{Y|X}(x,y)\,dy \\ &= -F_i(t)e^x - K + F_i^f(t)e^{\mu_y + \rho_i\sigma_y\frac{x-\mu_x}{\sigma_x} + \frac{1}{2}\sigma_y^2(1-\rho_i^2)} \end{aligned}$$

2. $F_i(t)e^x + K > 0$.

   Set $\bar{K} := F_i(t)e^x + K$ and $\bar{\omega} := -\omega$. Then

$$\begin{aligned} \left[\cdots\right] &= \int_{-\infty}^{+\infty} \left(\bar{\omega}F_i^f(t)e^y - \bar{\omega}K\right)^+ f_{Y|X}(x,y)\,dy \\ &= \bar{\omega}F_i^f(t)e^{\mu_y+\rho_i\sigma_y\frac{x-\mu_x}{\sigma_x}+\frac{1}{2}\sigma_y^2(1-\rho_i^2)}\Phi\left(\bar{\omega}\frac{\ln\frac{F_i^f(t)}{F_i(t)e^x+K} + \mu_y + \rho_i\sigma_y\frac{x-\mu_x}{\sigma_x} + \sigma_y^2(1-\rho_i^2)}{\sigma_y\sqrt{1-\rho_i^2}}\right) \\ &\quad - \bar{\omega}(F_i(t)e^x + K)\Phi\left(\bar{\omega}\frac{\ln\frac{F_i^f(t)}{F_i(t)e^x+K} + \mu_y + \rho_i\sigma_y\frac{x-\mu_x}{\sigma_x}}{\sigma_y\sqrt{1-\rho_i^2}}\right) \end{aligned}$$

   by formula (B.1) in Appendix B of Brigo and Mercurio (2001).

Finally, to obtain (5), we simply have to set $v := (x-\mu_x)/\sigma_x$. □

1.3 Options on the Product

The second example we consider is that of an option written on the product of the two Libor rates $L(T_{i-1}, T_i)$ and $L^f(T_{i-1}, T_i)$, whose payoff at time T_i, in domestic currency, is

$$\tau_i N \left[\omega\big(L(T_{i-1}, T_i)L^f(T_{i-1}, T_i) - K\big)\right]^+ = \tau_i N \left[\omega\Big(F_i(T_{i-1})F_i^f(T_{i-1}) - K\Big)\right]^+, \qquad (8)$$

where N is the nominal value, K is the strike price and $\omega = 1$ for a call and $\omega = -1$ for a put.

Proposition 1.2. *The no-arbitrage value at time t of the payoff (8) is given by*

$$\begin{aligned}
&\mathbf{LP}(t, T_{i-1}, T_i, \tau_i, N, K, \omega) = \tau_i N P(t, T_i) \\
&\cdot \Bigg[\omega F_i(t)F_i^f(t)e^{[-\rho\sigma_{F\mathcal{X}}\sigma_i^f + \rho_i\sigma_i\sigma_i^f]\tau}\Phi\left(\omega\frac{\ln\frac{F_i(t)F_i^f(t)}{K} + \big[2\rho_i\sigma_i\sigma_i^f - \rho\sigma_{F\mathcal{X}}\sigma_i^f + \frac{1}{2}\sigma_i^2 + \frac{1}{2}(\sigma_i^f)^2\big]\tau}{\sqrt{[\sigma_i^2 + (\sigma_i^f)^2 + 2\rho_i\sigma_i\sigma_i^f]\tau}}\right) \\
&\quad -\omega K\Phi\left(\omega\frac{\ln\frac{F_i(t)F_i^f(t)}{K} - \big[\rho\sigma_{F\mathcal{X}}\sigma_i^f + \frac{1}{2}\sigma_i^2 + \frac{1}{2}(\sigma_i^f)^2\big]\tau}{\sqrt{[\sigma_i^2 + (\sigma_i^f)^2 + 2\rho_i\sigma_i\sigma_i^f]\tau}}\right)\Bigg]
\end{aligned} \qquad (9)$$

Proof. Since

$$F_i(T_{i-1})F_i^f(T_{i-1}) = F_i(t)F_i^f(t)e^{-[\rho\sigma_{F\mathcal{X}}\sigma_i^f + \frac{1}{2}\sigma_i^2 + \frac{1}{2}(\sigma_i^f)^2]\tau + \sigma_i[W_i(T_{i-1}) - W_i(t)] + \sigma_i^f[W_i^f(T_{i-1}) - W_i^f(t)]},$$

we have that, under Q^i,

$$\begin{aligned}
&\ln\big[F_i(T_{i-1})F_i^f(T_{i-1})|\mathcal{F}_t\big] \sim \mathcal{N}(M, V^2), \\
&M = \ln\big[F_i(t)F_i^f(t)\big] - [\rho\sigma_{F\mathcal{X}}\sigma_i^f + \tfrac{1}{2}\sigma_i^2 + \tfrac{1}{2}(\sigma_i^f)^2]\tau, \\
&V = \sqrt{[\sigma_i^2 + (\sigma_i^f)^2 + 2\rho_i\sigma_i\sigma_i^f]\tau}.
\end{aligned}$$

To obtain (9), we simply have to remember (1) and apply formula (B.1) in Appendix B of Brigo and Mercurio (2001). □

1.4 Trigger swaps

The final example we consider is that of a swap where, in one leg, different payments are triggered by different levels of either the domestic or the foreign Libor rates.

In formulas, a leg of the trigger swap pays off at time T_i, in domestic currency, either

$$\tau_i N \left[\big(aF_i(T_{i-1}) + bF_i^f(T_{i-1}) + c\big)1_{\{\omega F_i(T_{i-1}) \geq \omega K\}}\right]\big(1 + \psi\tau_i F_i(T_{i-1})\big), \qquad (10)$$

or, in case the payment is triggered by the foreign rate,

$$\tau_i N \left[\big(aF_i(T_{i-1}) + bF_i^f(T_{i-1}) + c\big)1_{\{\omega F_i^f(T_{i-1}) \geq \omega K\}}\right]\big(1 + \psi\tau_i F_i(T_{i-1})\big), \qquad (11)$$

where N is the nominal value, a, b, c are real constants specified by the contract, ω is either 1 or -1, $\psi = 1$ for the "in-arrears" case and $\psi = 0$ otherwise.

Proposition 1.3. *The no-arbitrage value at time t of the payoff (10) is given by*

$$
\begin{aligned}
\mathbf{TSD}(t, T_{i-1}, T_i, \tau_i, N, K, \omega, \psi) &= \tau_i N P(t, T_i) \Bigg[(a + c\psi\tau_i) F_i(t) \Phi\Bigg(\omega \frac{\ln \frac{F_i(t)}{K} + \frac{1}{2}\sigma_i^2 \tau}{\sigma_i \sqrt{\tau}} \Bigg) \\
&+ a\psi\tau_i F_i^2(t) e^{\sigma_i^2 \tau} \Phi\Bigg(\omega \frac{\ln \frac{F_i(t)}{K} + \frac{3}{2}\sigma_i^2 \tau}{\sigma_i \sqrt{\tau}} \Bigg) + c\Phi\Bigg(\omega \frac{\ln \frac{F_i(t)}{K} - \frac{1}{2}\sigma_i^2 \tau}{\sigma_i \sqrt{\tau}} \Bigg) \\
&+ b F_i^f(t) e^{-\rho\sigma_{F\mathcal{X}}\sigma_i^f \tau} \Phi\Bigg(\omega \frac{\ln \frac{F_i(t)}{K} + \big[\rho_i \sigma_i \sigma_i^f - \frac{1}{2}\sigma_i^2\big]\tau}{\sigma_i \sqrt{\tau}} \Bigg) \\
&+ b\psi\tau_i F_i(t) F_i^f(t) e^{[-\rho\sigma_{F\mathcal{X}}\sigma_i^f + \rho_i\sigma_i\sigma_i^f]\tau} \Phi\Bigg(\omega \frac{\ln \frac{F_i(t)}{K} + \big[\rho_i \sigma_i \sigma_i^f + \frac{1}{2}\sigma_i^2\big]\tau}{\sigma_i \sqrt{\tau}} \Bigg) \Bigg].
\end{aligned}
\tag{12}
$$

The no-arbitrage value at time t of the payoff (11) is instead given by

$$
\begin{aligned}
\mathbf{TSF}(t, T_{i-1}, T_i, \tau_i, N, K, \omega, \psi) &= \tau_i N P(t, T_i) \Bigg[c\Phi\Bigg(\omega \frac{\ln \frac{F_i^f(t)}{K} - \big[\rho\sigma_{F\mathcal{X}} + \frac{1}{2}\sigma_i^f\big]\sigma_i^f \tau}{\sigma_i^f \sqrt{\tau}} \Bigg) \\
&+ (a + c\psi\tau_i) F_i(t) \Phi\Bigg(\omega \frac{\ln \frac{F_i^f(t)}{K} - \big[\rho\sigma_{F\mathcal{X}} + \frac{1}{2}\sigma_i^f - \rho_i\sigma_i\big]\sigma_i^f \tau}{\sigma_i^f \sqrt{\tau}} \Bigg) \\
&+ a\psi\tau_i F_i^2(t) e^{\sigma_i^2 \tau} \Phi\Bigg(\omega \frac{\ln \frac{F_i^f(t)}{K} - \big[\rho\sigma_{F\mathcal{X}} + \frac{1}{2}\sigma_i^f - 2\rho_i\sigma_i\big]\sigma_i^f \tau}{\sigma_i^f \sqrt{\tau}} \Bigg) \\
&+ b F_i^f(t) e^{-\rho\sigma_{F\mathcal{X}}\sigma_i^f \tau} \Phi\Bigg(\omega \frac{\ln \frac{F_i^f(t)}{K} + \big[-\rho\sigma_{F\mathcal{X}} + \frac{1}{2}\sigma_i^f\big]\sigma_i^f \tau}{\sigma_i^f \sqrt{\tau}} \Bigg) \\
&+ b\psi\tau_i F_i(t) F_i^f(t) e^{[-\rho\sigma_{F\mathcal{X}}\sigma_i^f + \rho_i\sigma_i\sigma_i^f]\tau} \Phi\Bigg(\omega \frac{\ln \frac{F_i^f(t)}{K} + \big[-\rho\sigma_{F\mathcal{X}} + \frac{1}{2}\sigma_i^f + \rho_i\sigma_i\big]\sigma_i^f \tau}{\sigma_i^f \sqrt{\tau}} \Bigg) \Bigg].
\end{aligned}
\tag{13}
$$

Proof. The proof is quite similar in spirit to that of Proposition 1.1 and is therefore omitted. The only difference is that here the outer integral, in both cases, can be explicitly calculated, too. □

References

[1] D. Brigo and F. Mercurio (2001). Interest Rate Models: Theory and Practice. Springer Finance, Heidelberg.