# PRICING INTEREST RATE AND INFLATION-LINKED HYBRIDS

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# Stylized facts

- Inflation-indexed bonds have been issued since the 80's, but it is only in the very last years that these bonds, and inflation-indexed derivatives in general, have become quite popular.

- Inflation is defined as the percentage increment of a reference index, the Consumer Price Index (CPI), which is a basket of good and services.

- Denoting by $I(t)$ the CPI's value at time $t$, the inflation rate over the time interval $[t, T]$ is therefore:

$$i(t, T) := \frac{I(T)}{I(t)} - 1.$$

- In theory, but also in practice, inflation can become negative.

# Stylized facts (cont'd)

## Historical plots of CPI's

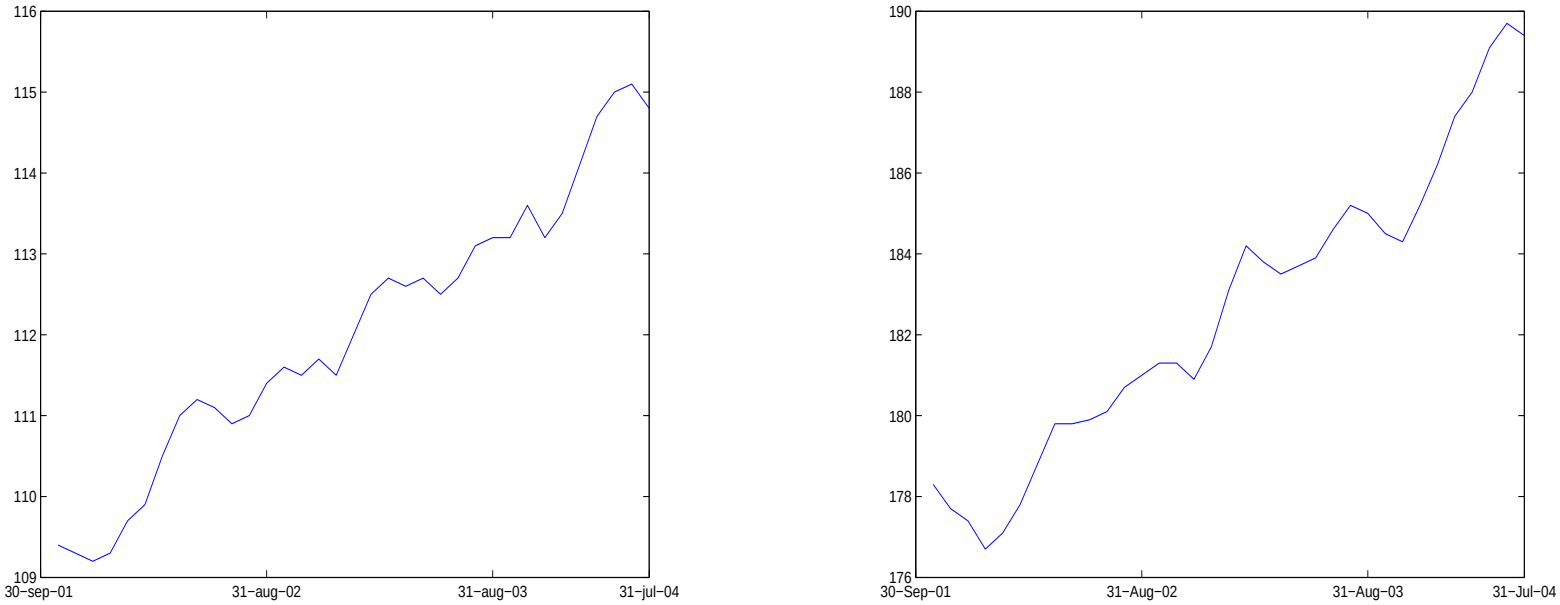



Figure 1: Left: EUR CPI Unrevised Ex-Tobacco. Right: USD CPI Urban Consumers NSA. Monthly closing values from 30-Sep-01 to 21-Jul-04.

# Stylized facts (cont'd)

- Banks are used to issue inflation-linked bonds, where a zero-strike floor is offered in conjunction with the "pure" bond.

- To grant positive coupons, the inflation rate is typically floored at zero.

- Accordingly, floors with low strikes are the most actively traded options on inflation rates.

- Other extremely popular derivatives are inflation-indexed swaps.

- Two are the main inflation-indexed swaps traded in the market:
  - the zero coupon (ZC) swap;
  - the year-on-year (YY) swap.

# The related literature

- Inflation-indexed derivatives require a specific model to be valued.
- Main references: Barone and Castagna (1997), van Bezooyen et al. (1997), Hughston (1998), Kazziha (1999), Cairns (2000), Jamshidian (2002), Jarrow and Yildirim (2003), Korn and Kruse (2003), Belgrade et al. (2004), Mercurio (2005), Kruse and Nögel (2006) and Mercurio and Moreni (2006).
- Inflation derivatives are priced with a foreign-currency analogy (the pricing is equivalent to that of a cross-currency interest-rate derivative).
- In a short rate approach, one models the evolution of the instantaneous nominal and real rates and of the CPI (interpreted as the "exchange rate" between the nominal and real economies).
- Recent approaches are based on market models, where one models forward CPI indices and nominal rates.

## Purpose and outline of the talk

- Our purpose is to price analytically, and consistently with no arbitrage, inflation-indexed swaps and options.

- We start by introducing the two main types of inflation swaps and price them by means of the JY model and two different market models.

- We then introduce inflation caps and floors, and derive analytical formulas under the JY model and the second ("flat smile") market model.

- We finally consider stochastic volatility as in Heston (1993) and derive closed-form formulas for caps and floors.

- Examples of calibration to market data are shown both in the "flat smile" case and in the stochastic volatility case.

## Some notations and definitions

We use the subscripts $n$ and $r$ to denote quantities in the nominal and real economies, respectively.

The zero-coupon bond prices at time $t$ for maturity $T$ in the nominal and real economies are denoted, respectively, by $P_n(t,T)$ and $P_r(t,T)$.

The instantaneous forward rates at time $t$ for maturity $T$ are defined by

$$f_x(t,T) = -\frac{\partial \ln P_x(t,T)}{\partial T}, \quad x \in \{n, r\}$$

and the corresponding instantaneous short rates by

$$n(t) = f_n(t,t),$$
$$r(t) = f_r(t,t).$$

## Some notations and definitions (cont'd)

Given the future time interval $[T_{i-1}, T_i]$, the related forward LIBOR rates, at time $t$, are

$$F_x(t; T_{i-1}, T_i) = \frac{P_x(t, T_{i-1}) - P_x(t, T_i)}{\tau_i P_x(t, T_i)}, \quad x \in \{n, r\},$$

The $T_i$-forward CPI is defined by

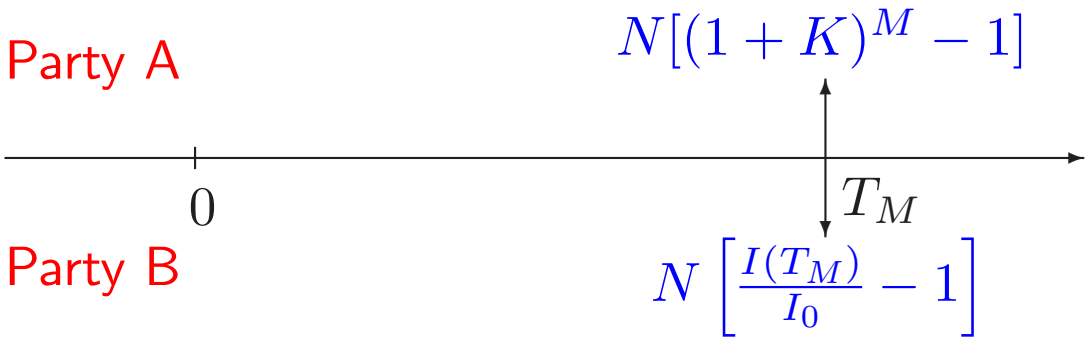
$$\mathcal{I}_i(t) := I(t) \frac{P_r(t, T_i)}{P_n(t, T_i)}.$$

We denote by $Q_n$ and $Q_r$ the nominal and real risk-neutral measures, respectively, and by $E_x$ the expectation associated to $Q_x$, $x \in \{n, r\}$.

We denote by $Q_n^T$ the nominal $T$-forward measure and by $E_n^T$ the associated expectation.

$\mathcal{F}_t$ denotes the $\sigma$-algebra generated by the relevant processes up to time $t$.

# Zero-coupon inflation-indexed swaps




In a ZCIIS, at time $T_M = M$ years, Party B pays Party A the fixed amount

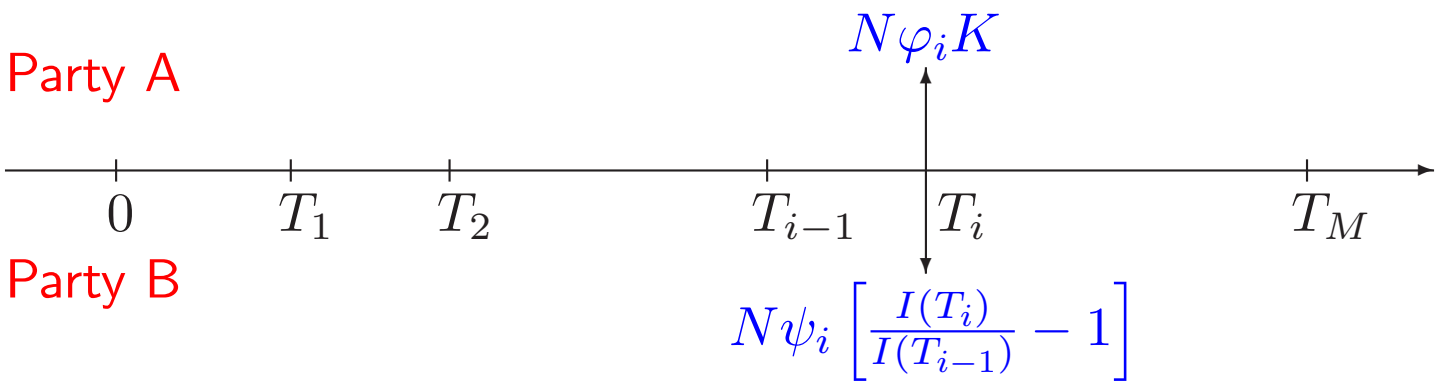
$$N[(1+K)^M - 1],$$

where $K$ and $N$ are, respectively, the contract fixed rate and the contract nominal value.

Party A pays Party B, at the final time $T_M$, the floating amount

$$N\left[\frac{I(T_M)}{I_0} - 1\right].$$

# Year-on-year inflation-indexed swaps




In a YYIIS, at each time $T_i$, Party B pays Party A the fixed amount

$$N\varphi_i K,$$

while Party A pays Party B the (floating) amount

$$N\psi_i \left[\frac{I(T_i)}{I(T_{i-1})} - 1\right],$$

where $\varphi_i$ and $\psi_i$ are, respectively, the fixed- and floating-leg year fractions for the interval $[T_{i-1}, T_i]$, $T_0 := 0$ and $N$ is again the swap nominal value.

# ZCIIS and YYIIS rates

Both ZC and YY swaps are quoted, in the market, in terms of the corresponding fixed rate $K$.

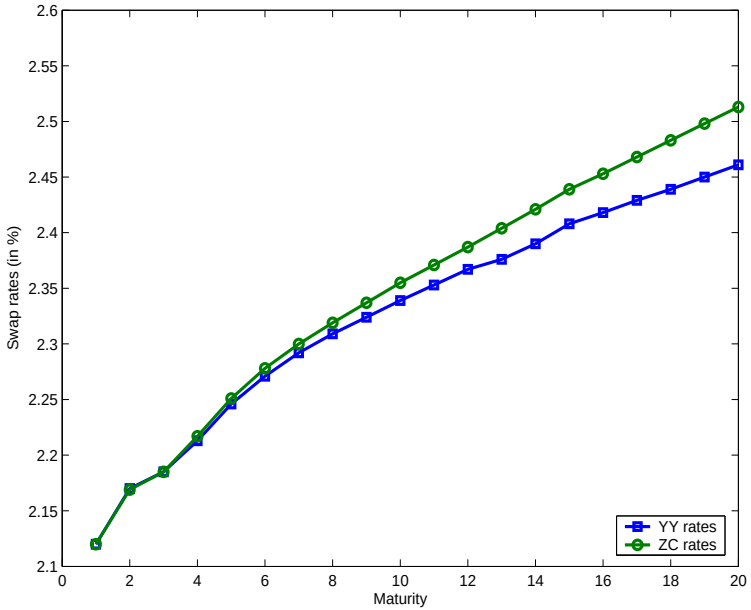



Figure 2: Euro inflation swap rates as of October 7, 2004. The reference CPI is the Euro-zone ex-tobacco index.

# Pricing of a ZCIIS

Standard risk-neutral pricing (under the nominal economy) implies that the value at time $t$, $0 \leq t < T_M$, of the inflation-indexed leg of the ZCIIS is

$$\textbf{ZCIIS}(t, T_M, I_0, N) = N E_n \left\{ e^{-\int_t^{T_M} n(u)\,du} \left[ \frac{I(T_M)}{I_0} - 1 \right] \middle| \mathcal{F}_t \right\}.$$

The foreign-currency analogy implies that, for each $t < T$:

| | time $T$ | | time $t$ |
|---|---|---|---|
| Nominal | $I(T)$ | $\cdots$ | $E_n\left\{ e^{-\int_t^T n(u)\,du} I(T) \middle| \mathcal{F}_t \right\}$ |
| Real | $1$ | | $P_r(t,T)$ |

$$\Rightarrow \; E_n \left\{ e^{-\int_t^T n(u)\,du} I(T) \middle| \mathcal{F}_t \right\} = I(t) P_r(t,T) \Rightarrow$$

$$\textbf{ZCIIS}(t, T_M, I_0, N) = N \left[ \frac{I(t)}{I_0} P_r(t, T_M) - P_n(t, T_M) \right] = N P_n(t, T_M) \left[ \frac{\mathcal{I}_M(t)}{I_0} - 1 \right]$$

# Pricing of a ZCIIS (cont'd)

The ZCIIS price is therefore model-independent: it is not based on specific assumptions on the interest rates evolution, but simply follows from the absence of arbitrage.

This result is extremely important since it enables us to strip, with no ambiguity, real zero-coupon bond prices (equivalently, forward CPI's) from the quoted prices of zero-coupon inflation-indexed swaps.

The market quotes values of $K = K(T_M)$ for some given maturities $T_M$.

The ZCIIS corresponding to $(T_M, K(T_M))$ has zero value at time $t = 0$ if and only if

$$N[P_r(0, T_M) - P_n(0, T_M)] = NP_n(0, T_M)[(1 + K(T_M))^M - 1]$$

$$\Rightarrow P_r(0, T_M) = P_n(0, T_M)(1 + K(T_M))^M \Rightarrow \mathcal{I}_M(0) = I(0)(1 + K(T_M))^M$$

# Pricing of a YYIIS

The valuation of a YYIIS is less straightforward and, as we shall see, requires the specification of an interest rate model.

The value at time $t < T_i$ of the YYIIS payoff at time $T_i$ is

$$\textbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i E_n \left\{ e^{-\int_t^{T_i} n(u)\,du} \left[ \frac{I(T_i)}{I(T_{i-1})} - 1 \right] \Big| \mathcal{F}_t \right\},$$

which, assuming $t < T_{i-1}$, can be calculated as

$$N\psi_i E_n \left\{ e^{-\int_t^{T_{i-1}} n(u)\,du} E_n \left[ e^{-\int_{T_{i-1}}^{T_i} n(u)\,du} \left( \frac{I(T_i)}{I(T_{i-1})} - 1 \right) \Big| \mathcal{F}_{T_{i-1}} \right] \Big| \mathcal{F}_t \right\}$$
$$= N\psi_i E_n \left\{ e^{-\int_t^{T_{i-1}} n(u)\,du} [P_r(T_{i-1}, T_i) - P_n(T_{i-1}, T_i)] \Big| \mathcal{F}_t \right\}.$$

This expectation is, in general, model dependent.

# The JY model

Under the real-world probability space $(\Omega, \mathcal{F}, P)$, with associated filtration $\mathcal{F}_t$, Jarrow and Yildirim assumed that

$$
\begin{aligned}
df_n(t,T) &= \alpha_n(t,T)\,dt + \varsigma_n(t,T)\,dW_n^P(t) \\
df_r(t,T) &= \alpha_r(t,T)\,dt + \varsigma_r(t,T)\,dW_r^P(t) \\
dI(t) &= I(t)\mu(t)\,dt + \sigma_I I(t)\,dW_I^P(t)
\end{aligned}
$$

with $I(0) = I_0 > 0$, and $f_x(0,T) = f_x^{\mathrm{MKT}}(0,T)$, $x \in \{n, r\}$, where

- $(W_n^P, W_r^P, W_I^P)$ is a Brownian motion with correlations $\rho_{n,r}$, $\rho_{n,I}$ and $\rho_{r,I}$;
- $\alpha_n$, $\alpha_r$ and $\mu$ are adapted processes;
- $\varsigma_n$ and $\varsigma_r$ are deterministic functions;
- $\sigma_I$ is a positive constant.

# The JY model (cont'd)

Choosing the forward rate volatilities as

$$\varsigma_n(t,T) = \sigma_n\, e^{-a_n(T-t)}, \quad \varsigma_r(t,T) = \sigma_r\, e^{-a_r(T-t)},$$

where $\sigma_n$ $\sigma_r$, $a_n$ and $a_r$ are positive constants, and using the equivalent formulation in terms of instantaneous short rates, we have the following.

**Proposition.** The $Q_n$-dynamics of $n$, $r$ and $I$ are, respectively,

$$\begin{aligned} dn(t) &= [\vartheta_n(t) - a_n n(t)]\, dt + \sigma_n\, dW_n(t) \\ dr(t) &= [\vartheta_r(t) - \rho_{r,I}\sigma_I\sigma_r - a_r r(t)]\, dt + \sigma_r\, dW_r(t) \\ dI(t) &= I(t)[n(t) - r(t)]\, dt + \sigma_I I(t)\, dW_I(t) \end{aligned}$$

where $(W_n, W_r, W_I)$ is a Brownian motion with correlations $\rho_{n,r}$, $\rho_{n,I}$ and $\rho_{r,I}$, and

$$\vartheta_x(t) = \frac{\partial f_x(0,t)}{\partial T} + a_x f_x(0,t) + \frac{\sigma_x^2}{2a_x}(1 - e^{-2a_x t}), \quad x \in \{n, r\}.$$

# Pricing of a YYIIS: the JY model

We remember that:

$$\begin{aligned}\textbf{YYIIS}(t) &= N\psi_i E_n\left\{e^{-\int_t^{T_{i-1}} n(u)\,du} P_r(T_{i-1},T_i)\Big|\mathcal{F}_t\right\} - N\psi_i P_n(t,T_i)\\ &= N\psi_i P_n(t,T_{i-1}) E_n^{T_{i-1}}\left\{P_r(T_{i-1},T_i)\Big|\mathcal{F}_t\right\} - N\psi_i P_n(t,T_i),\end{aligned}$$

and also remember the zero-coupon bond price formula in the Hull and White (1994) model:

$$\begin{aligned}P_r(t,T) &= A_r(t,T)e^{-B_r(t,T)r(t)},\\ B_r(t,T) &= \frac{1}{a_r}\left[1-e^{-a_r(T-t)}\right],\\ A_r(t,T) &= \frac{P_r^M(0,T)}{P_r^M(0,t)}\exp\left\{B_r(t,T)f_r^M(0,t) - \frac{\sigma^2}{4a_r}(1-e^{-2a_r t})B_r(t,T)^2\right\}.\end{aligned}$$

# Pricing of a YYIIS: the JY model (cont'd)

Since the real instantaneous rate evolves under $Q_n^{T_{i-1}}$ according to

$$dr(t) = [-\rho_{n,r}\sigma_n\sigma_r B_n(t,T_{i-1}) + \vartheta_r(t) - \rho_{r,I}\sigma_I\sigma_r - a_r r(t)]\,dt + \sigma_r\,dW_r^{T_{i-1}}(t),$$

$r(T_{i-1})$ remains a normal random variable, and hence the real bond price $P_r(T_{i-1},T_i)$ is lognormally distributed also under $Q_n^{T_{i-1}}$.

After some tedious, but straightforward, algebra we finally obtain

$$\textbf{YYIIS}(t) = N\psi_i P_n(t,T_{i-1})\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{C(t,T_{i-1},T_i)} - N\psi_i P_n(t,T_i),$$

where

$$C(t,T_{i-1},T_i) = \sigma_r B_r(T_{i-1},T_i)\Bigg[B_r(t,T_{i-1})\Big(\rho_{r,I}\sigma_I - \tfrac{1}{2}\sigma_r B_r(t,T_{i-1}) + \frac{\rho_{n,r}\sigma_n}{a_n+a_r}\big(1 + a_r B_n(t,T_{i-1})\big)\Big) - \frac{\rho_{n,r}\sigma_n}{a_n+a_r}B_n(t,T_{i-1})\Bigg].$$

# Pricing of a YYIIS: the JY model (cont'd)

The value at time $t$ of the inflation-indexed leg of the swap is simply obtained by summing up the values of all floating payments. We thus get

$$\textbf{YYIIS}(t,\mathcal{T},\Psi,N) = N\psi_{\iota(t)}\left[\frac{I(t)}{I(T_{\iota(t)-1})}P_r(t,T_{\iota(t)}) - P_n(t,T_{\iota(t)})\right]$$

$$+ N\sum_{i=\iota(t)+1}^{M}\psi_i\left[P_n(t,T_{i-1})\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}\,e^{C(t,T_{i-1},T_i)} - P_n(t,T_i)\right],$$

where $\mathcal{T}=\{T_1,...,T_M\}$, $\Psi=\{\psi_1,...,\psi_M\}$ and $\iota(t)=\min\{i:T_i>t\}$. At $t=0$:

$$\textbf{YYIIS}(0) = N\sum_{i=1}^{M}\psi_i P_n(0,T_i)\left[\frac{\mathcal{I}_i(0)}{\mathcal{I}_{i-1}(0)}\,e^{C(0,T_{i-1},T_i)} - 1\right]$$

$$= N\sum_{i=1}^{M}\psi_i P_n(0,T_i)\left[\frac{1+\tau_i F_n(0;T_{i-1},T_i)}{1+\tau_i F_r(0;T_{i-1},T_i)}\,e^{C(0,T_{i-1},T_i)} - 1\right].$$

# Pricing of a YYIIS: the LIBOR market model (MM1)

For an alternative pricing of the above YYIIS, we notice that

$$P_n(t,T_{i-1})E_n^{T_{i-1}}\left\{P_r(T_{i-1},T_i)\big|\mathcal{F}_t\right\} = P_n(t,T_i)E_n^{T_i}\left\{\frac{P_r(T_{i-1},T_i)}{P_n(T_{i-1},T_i)}\big|\mathcal{F}_t\right\}$$

$$= P_n(t,T_i)E_n^{T_i}\left\{\frac{1+\tau_i F_n(T_{i-1};T_{i-1},T_i)}{1+\tau_i F_r(T_{i-1};T_{i-1},T_i)}\big|\mathcal{F}_t\right\}.$$

It seems natural, therefore, to resort to a (lognormal) LIBOR model, for both nominal and real rates, see Mercurio (2005).

Since $I(t)P_r(t,T_i)$ is the price of an asset in the nominal economy, the forward CPI

$$\mathcal{I}_i(t) = I(t)\frac{P_r(t,T_i)}{P_n(t,T_i)}$$

is a martingale under $Q_n^{T_i}$.

# Pricing of a YYIIS: the LIBOR market model (cont'd)

We assume:

$$d\mathcal{I}_i(t) = \sigma_{I,i}\mathcal{I}_i(t)\, dW_i^I(t).$$

where $\sigma_{I,i}$ is a positive constant and $W_i^I$ is a $Q_n^{T_i}$-Brownian motion.

Assuming also that both nominal and real forward rates follow a lognormal LIBOR market model, the foreign-currency analogy implies that, under $Q_n^{T_i}$,

$$\begin{aligned} dF_n(t;T_{i-1},T_i) &= \sigma_{n,i}F_n(t;T_{i-1},T_i)\, dW_i^n(t), \\ dF_r(t;T_{i-1},T_i) &= F_r(t;T_{i-1},T_i)\big[-\rho_{I,r,i}\sigma_{I,i}\sigma_{r,i}\, dt + \sigma_{r,i}\, dW_i^r(t)\big], \end{aligned}$$

where $\sigma_{n,i}$ and $\sigma_{r,i}$ are positive constants, $W_i^n$ and $W_i^r$ are two standard Brownian motions with instantaneous correlation $\rho_i$, and $\rho_{I,r,i}$ is the instantaneous correlation between $\mathcal{I}_i(\cdot)$ and $F_r(\cdot;T_{i-1},T_i)$.

# Pricing of a YYIIS: the LIBOR market model (cont'd)

The last expectation can now be easily calculated with a numerical integration by noting that, under $Q_n^{T_i}$ and conditional on $\mathcal{F}_t$, the pair

$$(X_i, Y_i) = \left( \ln \frac{F_n(T_{i-1}; T_{i-1}, T_i)}{F_n(t; T_{i-1}, T_i)}, \ln \frac{F_r(T_{i-1}; T_{i-1}, T_i)}{F_r(t; T_{i-1}, T_i)} \right)$$

is distributed as a bivariate normal random variable with mean vector and variance-covariance matrix respectively given by

$$M_{X_i,Y_i} = \begin{bmatrix} \mu_{x,i}(t) \\ \mu_{y,i}(t) \end{bmatrix}, \quad V_{X_i,Y_i} = \begin{bmatrix} \sigma_{x,i}^2(t) & \rho_i \sigma_{x,i}(t) \sigma_{y,i}(t) \\ \rho_i \sigma_{x,i}(t) \sigma_{y,i}(t) & \sigma_{y,i}^2(t) \end{bmatrix},$$

where

$$\mu_{x,i}(t) = -\tfrac{1}{2}\sigma_{n,i}^2 (T_{i-1} - t), \quad \sigma_{x,i}(t) = \sigma_{n,i}\sqrt{T_{i-1} - t},$$

$$\mu_{y,i}(t) = \left[ -\tfrac{1}{2}\sigma_{r,i}^2 - \rho_{I,r,i}\sigma_{I,i}\sigma_{r,i} \right] (T_{i-1} - t), \quad \sigma_{y,i}(t) = \sigma_{r,i}\sqrt{T_{i-1} - t}.$$

# Pricing of a YYIIS: the LIBOR market model (cont'd)

It is well known that the joint density $f_{X_i,Y_i}(x,y)$ can be decomposed as

$$f_{X_i,Y_i}(x,y) = f_{X_i|Y_i}(x,y) f_{Y_i}(y),$$

where

$$f_{X_i|Y_i}(x,y) = \frac{1}{\sigma_{x,i}(t)\sqrt{2\pi}\sqrt{1-\rho_i^2}} \exp\left[ -\frac{\left(\frac{x-\mu_{x,i}(t)}{\sigma_{x,i}(t)} - \rho_i \frac{y-\mu_{y,i}(t)}{\sigma_{y,i}(t)}\right)^2}{2(1-\rho_i^2)} \right]$$

$$f_{Y_i}(y) = \frac{1}{\sigma_{y,i}(t)\sqrt{2\pi}} \exp\left[ -\frac{1}{2}\left(\frac{y-\mu_{y,i}(t)}{\sigma_{y,i}(t)}\right)^2 \right].$$

The expectation can thus be calculated as (we set $F_x(t) := F_x(t; T_{i-1}, T_i)$)

$$\int_{-\infty}^{+\infty} \frac{1}{1+\tau_i F_r(t)\, e^y} \left[ \int_{-\infty}^{+\infty} \left(1+\tau_i F_n(t) e^x\right) f_{X_i|Y_i}(x,y)\, dx \right] f_{Y_i}(y)\, dy$$

## Pricing of a YYIIS: the LIBOR market model (cont'd)

The value at time $t$ of the inflation-indexed leg of the swap is thus given by

$$\textbf{YYIIS}(t,\mathcal{T},\Psi,N) = N\psi_{\iota(t)}\left[\frac{I(t)}{I(T_{\iota(t)-1})}P_r(t,T_{\iota(t)}) - P_n(t,T_{\iota(t)})\right]$$

$$+ N\sum_{i=\iota(t)+1}^{M}\psi_i P_n(t,T_i)\left[\int_{-\infty}^{+\infty}\frac{1+\tau_i F_n(t)\,e^{\rho_i\sigma_{x,i}(t)z-\frac{1}{2}\sigma_{x,i}^2(t)\rho_i^2}}{1+\tau_i F_r(t)\,e^{\mu_{y,i}(t)+\sigma_{y,i}(t)z}}\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\,dz - 1\right].$$

**N.B.** In theory, the volatilities $\sigma_{I,i}$ can not be constant for each $i$, see Schlögl (2002). In practice, however, they are approximately constant.

In particular at $t=0$, $\textbf{YYIIS}(0,\mathcal{T},\Psi,N) =$

$$= N\sum_{i=1}^{M}\psi_i P_n(0,T_i)\left[\int_{-\infty}^{+\infty}\frac{1+\tau_i F_n(0)\,e^{\rho_i\sigma_{x,i}(0)z-\frac{1}{2}\sigma_{x,i}^2(0)\rho_i^2}}{1+\tau_i F_r(0)\,e^{\mu_{y,i}(0)+\sigma_{y,i}(0)z}}\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\,dz - 1\right]$$

# Pricing of a YYIIS: the LIBOR market model (cont'd)

This YYIIS price depends on: the (instantaneous) volatilities of nominal and real forward rates and their correlations, for $i = 2, ..., M$; the correlations between real forward rates and forward inflation indices, again for $i = 2, ..., M$.

Compared with the JY expression, this last formula looks more complicated both in terms of input parameters and in terms of the calculations involved.

However, one-dimensional numerical integrations are not so cumbersome. Moreover, as is typical in a market model, the input parameters can be determined more easily than in the previous short-rate approach.

Both approaches seen so far have the drawback that the volatility of real rates may be hard to estimate (historically). This estimation issue is overcome by a different market model, first proposed by Kazziha (1999), which is now described.

# Pricing of a YYIIS: a second market model (MM2)

Applying the definition of forward CPI and using the fact that $\mathcal{I}_i$ is a martingale under $Q_n^{T_i}$, we can also write, for $t < T_{i-1}$,

$$\begin{aligned}
\textbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) &= N\psi_i P(t, T_i) E_n^{T_i}\left\{\frac{I(T_i)}{I(T_{i-1})} - 1\middle|\mathcal{F}_t\right\} \\
&= N\psi_i P(t, T_i) E_n^{T_i}\left\{\frac{\mathcal{I}_i(T_i)}{\mathcal{I}_{i-1}(T_{i-1})} - 1\middle|\mathcal{F}_t\right\} \\
&= N\psi_i P(t, T_i) E_n^{T_i}\left\{\frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} - 1\middle|\mathcal{F}_t\right\}.
\end{aligned}$$

We recall that, under $Q_n^{T_i}$,

$$d\mathcal{I}_i(t) = \sigma_{I,i}\mathcal{I}_i(t)\, dW_i^I(t)$$

and that an analogous evolution holds for $\mathcal{I}_{i-1}$ under $Q_n^{T_{i-1}}$.

# Pricing of a YYIIS: a second market model (cont'd)

The dynamics of $\mathcal{I}_{i-1}$ under $Q_n^{T_i}$ are

$$d\mathcal{I}_{i-1}(t) = -\mathcal{I}_{i-1}(t)\sigma_{I,i-1}\frac{\tau_i\sigma_{n,i}F_n(t;T_{i-1},T_i)}{1+\tau_i F_n(t;T_{i-1},T_i)}\rho_{I,n,i}\,dt$$
$$+\,\sigma_{I,i-1}\mathcal{I}_{i-1}(t)\,dW_{i-1}^I(t),$$

where $\sigma_{I,i-1}$ is a positive constant, $W_{i-1}^I$ is a $Q_n^{T_i}$-Brownian motion with $dW_{i-1}^I(t)\,dW_i^I(t) = \rho_{I,i}\,dt$, and $\rho_{I,n,i}$ is the instantaneous correlation between $\mathcal{I}_{i-1}(\cdot)$ and $F_n(\cdot;T_{i-1},T_i)$.

The evolution of $\mathcal{I}_{i-1}$, under $Q_n^{T_i}$, depends on the nominal rate $F_n(\cdot;T_{i-1},T_i)$.

To avoid unpleasant calculations, we freeze the above drift at its current time-$t$ value, so that $\mathcal{I}_{i-1}(T_{i-1})$ conditional on $\mathcal{F}_t$ is lognormally distributed also under $Q_n^{T_i}$.

## Pricing of a YYIIS: a second market model (cont'd)

Also the ratio $\mathcal{I}_i(T_{i-1})/\mathcal{I}_{i-1}(T_{i-1})$ conditional on $\mathcal{F}_t$ is lognormally distributed under $Q_n^{T_i}$. This leads to

$$E_n^{T_i}\left\{\frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})}\Big|\mathcal{F}_t\right\} = \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}\,e^{D_i(t)},$$

where

$$D_i(t) = \sigma_{I,i-1}\left[\frac{\tau_i\sigma_{n,i}F_n(t;T_{i-1},T_i)}{1+\tau_i F_n(t;T_{i-1},T_i)}\rho_{I,n,i} - \rho_{I,i}\sigma_{I,i} + \sigma_{I,i-1}\right](T_{i-1}-t),$$

so that

$$\begin{aligned}\textbf{YYIIS}(t,T_{i-1},T_i,\psi_i,N) &= N\psi_i P_n(t,T_i)\left[\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)}\,e^{D_i(t)} - 1\right]\\ &= N\psi_i P_n(t,T_i)\left[\frac{P_n(t,T_{i-1})P_r(t,T_i)}{P_n(t,T_i)P_r(t,T_{i-1})}\,e^{D_i(t)} - 1\right]\end{aligned}$$

## Pricing of a YYIIS: a second market model (cont'd)

Finally, the value at time $t$ of the inflation-indexed leg of the swap is

$$\textbf{YYIIS}(t, \mathcal{T}, \Psi, N) = N\psi_{\iota(t)}\left[\frac{I(t)}{I(T_{\iota(t)-1})}P_r(t, T_{\iota(t)}) - P_n(t, T_{\iota(t)})\right]$$

$$+ N\sum_{i=\iota(t)+1}^{M}\psi_i\left[P_n(t, T_{i-1})\frac{P_r(t, T_i)}{P_r(t, T_{i-1})}e^{D_i(t)} - P_n(t, T_i)\right].$$

In particular at $t = 0$,

$$\begin{aligned}\textbf{YYIIS}(0, \mathcal{T}, \Psi, N) &= N\sum_{i=1}^{M}\psi_i P_n(0, T_i)\left[\frac{\mathcal{I}_i(0)}{\mathcal{I}_{i-1}(0)}e^{D_i(0)} - 1\right] \\ &= N\sum_{i=1}^{M}\psi_i P_n(0, T_i)\left[\frac{1 + \tau_i F_n(0; T_{i-1}, T_i)}{1 + \tau_i F_r(0; T_{i-1}, T_i)}e^{D_i(0)} - 1\right]\end{aligned}$$

# Pricing of a YYIIS: a second market model (cont'd)

This YYIIS price depends on: the (instantaneous) volatilities of forward inflation indices and their correlations; the (instantaneous) volatilities of nominal forward rates; the instantaneous correlations between forward inflation indices and nominal forward rates.

This pricing formula looks pretty similar to that in the JY case and may be preferred to the one in the LIBOR market model, since it combines the advantage of a fully-analytical formula with that of a market-model approach.

Moreover, the correction term $D$ does not depend on the volatility of real rates.

The only drawback is that the approximation it is based on may be rough for long maturities $T_i$. The formula, however, is exact when the correlations $\rho_{I,n,i}$ are set to zero and the terms $D_i$ are simplified accordingly.

# Inflation-indexed caplets

An Inflation-Indexed Caplet (IIC) is a call option on the inflation rate implied by the CPI index.

Analogously, an Inflation-Indexed Floorlet (IIF) is a put option on the same inflation rate.

In formulas, at time $T_i$, the IICF payoff is

$$N\psi_i \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa\right)\right]^+,$$

where $\kappa$ is the IICF strike, $\psi_i$ is the contract year fraction for the interval $[T_{i-1}, T_i]$, $N$ is the contract nominal value, and $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet.

We set $K := 1 + \kappa$.

# Inflation-indexed caplets (cont'd)

Standard no-arbitrage pricing theory implies that the value at time $t \leq T_{i-1}$ of the IICF at time $T_i$ is

$$\begin{aligned}
&\textbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) \\
&= N\psi_i E_n \left\{ e^{-\int_t^{T_i} n(u)\,du} \left[ \omega \left( \frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \Big| \mathcal{F}_t \right\} \\
&= N\psi_i P_n(t, T_i) E_n^{T_i} \left\{ \left[ \omega \left( \frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \Big| \mathcal{F}_t \right\}.
\end{aligned}$$

The pricing of a IICF is thus similar to that of a forward-start (cliquet) option.

We now provide analytical formulas both under the JY model and under the second market model.

# Inflation-indexed caplets: the JY model

The assumption of Gaussian nominal and real rates leads to a CPI that is lognomally distributed under $Q_n$.

When we move to a (nominal) forward measure the type of distribution is preserved.

Hence, $\frac{I(T_i)}{I(T_{i-1})}$ conditional on $\mathcal{F}_t$ is lognormally distributed also under $Q_n^{T_i}$, and the IICF price can be calculated as follows.

If $X$ is a lognormal random variable with $E(X) = m$ and Std$[\ln(X)] = v$, then

$$E\left\{[\omega(X-K)]^+\right\} = \omega m \Phi\left(\omega \frac{\ln \frac{m}{K} + \frac{1}{2}v^2}{v}\right) - \omega K \Phi\left(\omega \frac{\ln \frac{m}{K} - \frac{1}{2}v^2}{v}\right),$$

where $\Phi$ denotes the standard normal distribution function.

# Inflation-indexed caplets: the JY model (cont'd)

The conditional expectation of $I(T_i)/I(T_{i-1})$ is immediately obtained through the price of a YYIIS:

$$E_n^{T_i}\left\{\frac{I(T_i)}{I(T_{i-1})}\Big|\mathcal{F}_t\right\} = \frac{P_n(t,T_{i-1})}{P_n(t,T_i)}\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}\,e^{C(t,T_{i-1},T_i)}.$$

The variance of the log of the ratio can be equivalently calculated under the (nominal) risk-neutral measure. We get:

$$\mathrm{Var}_n^{T_i}\left\{\ln\frac{I(T_i)}{I(T_{i-1})}\Big|\mathcal{F}_t\right\} = V^2(t,T_{i-1},T_i),$$

where, setting $\Delta T_i := T_i - T_{i-1}$,

$$V^2(t,T_{i-1},T_i) =$$

# Inflation-indexed caplets: the JY model (cont'd)

$$= \frac{\sigma_n^2}{2a_n^3}(1-e^{-a_n\Delta T_i})^2[1-e^{-2a_n(T_{i-1}-t)}] + \frac{\sigma_r^2}{2a_r^3}(1-e^{-a_r\Delta T_i})^2[1-e^{-2a_r(T_{i-1}-t)}]$$

$$- 2\rho_{n,r}\frac{\sigma_n\sigma_r}{a_n a_r(a_n+a_r)}(1-e^{-a_n\Delta T_i})(1-e^{-a_r\Delta T_i})[1-e^{-(a_n+a_r)(T_{i-1}-t)}]$$

$$+ \sigma_I^2\Delta T_i + \frac{\sigma_n^2}{a_n^2}\left[\Delta T_i + \frac{2}{a_n}e^{-a_n\Delta T_i} - \frac{1}{2a_n}e^{-2a_n\Delta T_i} - \frac{3}{2a_n}\right]$$

$$+ \frac{\sigma_r^2}{a_r^2}\left[\Delta T_i + \frac{2}{a_r}e^{-a_r\Delta T_i} - \frac{1}{2a_r}e^{-2a_r\Delta T_i} - \frac{3}{2a_r}\right]$$

$$- 2\rho_{n,r}\frac{\sigma_n\sigma_r}{a_n a_r}\left[\Delta T_i - \frac{1-e^{-a_n\Delta T_i}}{a_n} - \frac{1-e^{-a_r\Delta T_i}}{a_r} + \frac{1-e^{-(a_n+a_r)\Delta T_i}}{a_n+a_r}\right]$$

$$+ 2\rho_{n,I}\frac{\sigma_n\sigma_I}{a_n}\left[\Delta T_i - \frac{1-e^{-a_n\Delta T_i}}{a_n}\right] - 2\rho_{r,I}\frac{\sigma_r\sigma_I}{a_r}\left[\Delta T_i - \frac{1-e^{-a_r\Delta T_i}}{a_r}\right].$$

## Inflation-indexed caplets: the JY model (cont'd)

We finally have:

$$\textbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega)$$

$$= \omega N \psi_i P_n(t, T_i) \left[ \frac{P_n(t, T_{i-1})}{P_n(t, T_i)} \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} e^{C(t, T_{i-1}, T_i)} \right.$$

$$\cdot \Phi \left( \omega \frac{\ln \frac{P_n(t,T_{i-1}) P_r(t,T_i)}{K P_n(t,T_i) P_r(t,T_{i-1})} + C(t, T_{i-1}, T_i) + \frac{1}{2} V^2(t, T_{i-1}, T_i)}{V(t, T_{i-1}, T_i)} \right)$$

$$\left. - K \Phi \left( \omega \frac{\ln \frac{P_n(t,T_{i-1}) P_r(t,T_i)}{K P_n(t,T_i) P_r(t,T_{i-1})} + C(t, T_{i-1}, T_i) - \frac{1}{2} V^2(t, T_{i-1}, T_i)}{V(t, T_{i-1}, T_i)} \right) \right],$$

which is clearly of a Black and Scholes type.

# Inflation-indexed caplets: a market model

We now try and calculate the IICF price under a market model. To this end, we apply the tower property of conditional expectations to get

$$\textbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega)$$

$$= N\psi_i P_n(t, T_i) E_n^{T_i} \left\{ \frac{E_n^{T_i}\left\{ [\omega(I(T_i) - KI(T_{i-1}))]^+ \,|\mathcal{F}_{T_{i-1}} \right\}}{I(T_{i-1})} \Big| \mathcal{F}_t \right\},$$

where we assume that $I(T_{i-1}) > 0$.

Sticking to a market-model approach, the calculation of the outer expectation depends on whether we model forward rates or the forward CPIs.

We here follow the second market model, since it allows the derivation of a simpler formula with less input parameters.

## Inflation-indexed caplets: a market model (cont'd)

Assuming again that, under $Q_n^{T_i}$,

$$d\mathcal{I}_i(t) = \sigma_{I,i}\mathcal{I}_i(t)\, dW_i^I(t)$$

and remembering that $I(T_i) = \mathcal{I}_i(T_i)$, we have:

$$\begin{aligned}
&E_n^{T_i}\Big\{ \left[\omega(I(T_i) - KI(T_{i-1}))\right]^+ |\mathcal{F}_{T_{i-1}}\Big\} \\
&= E_n^{T_i}\Big\{ \left[\omega(\mathcal{I}_i(T_i) - KI(T_{i-1}))\right]^+ |\mathcal{F}_{T_{i-1}}\Big\} \\
&= \omega\mathcal{I}_i(T_{i-1})\Phi\left(\omega\frac{\ln\frac{\mathcal{I}_i(T_{i-1})}{KI(T_{i-1})} + \frac{1}{2}\sigma_{I,i}^2(T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}}\right) \\
&\quad - \omega KI(T_{i-1})\Phi\left(\omega\frac{\ln\frac{\mathcal{I}_i(T_{i-1})}{KI(T_{i-1})} - \frac{1}{2}\sigma_{I,i}^2(T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}}\right).
\end{aligned}$$

## Inflation-indexed caplets: a market model (cont'd)

By definition of $\mathcal{I}_{i-1}$, the IICF price thus becomes

$$\omega N \psi_i P_n(t,T_i) E_n^{T_i} \left\{ \frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} \Phi\left( \omega \frac{\ln \frac{\mathcal{I}_i(T_{i-1})}{K\mathcal{I}_{i-1}(T_{i-1})} + \frac{1}{2}\sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}} \right) \right.$$
$$\left. -K\Phi\left( \omega \frac{\ln \frac{\mathcal{I}_i(T_{i-1})}{K\mathcal{I}_{i-1}(T_{i-1})} - \frac{1}{2}\sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i}\sqrt{T_i - T_{i-1}}} \right) \Big| \mathcal{F}_t \right\}.$$

Remembering the dynamics of $\mathcal{I}_{i-1}$ under $Q_n^{T_i}$, and freezing again the drift at its time-$t$ value, we have that under $Q_n^{T_i}$:

$$\ln \frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} | \mathcal{F}_t \sim \mathcal{N}\left( \ln \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} + D_i(t) - \frac{1}{2} V_i^2(t), V_i^2(t) \right).$$

where $V_i^2(t) := (\sigma_{I,i-1}^2 + \sigma_{I,i}^2 - 2\rho_{I,i}\sigma_{I,i-1}\sigma_{I,i})(T_{i-1} - t)$. Therefore,

## Inflation-indexed caplets: a market model (cont'd)

$$\textbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega)$$

$$= \omega N \psi_i P_n(t, T_i) \left[ \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} e^{D_i(t)} \Phi\left( \omega \frac{\ln \frac{\mathcal{I}_i(t)}{K\mathcal{I}_{i-1}(t)} + D_i(t) + \frac{1}{2}\mathcal{V}_i^2(t)}{\mathcal{V}_i(t)} \right) \right.$$

$$\left. - K\Phi\left( \omega \frac{\ln \frac{\mathcal{I}_i(t)}{K\mathcal{I}_{i-1}(t)} + D_i(t) - \frac{1}{2}\mathcal{V}_i^2(t)}{\mathcal{V}_i(t)} \right) \right],$$

where $\mathcal{V}_i(t) := \sqrt{V_i^2(t) + \sigma_{I,i}^2(T_i - T_{i-1})}$.

This price depends on the volatilities of the two forward inflation indices and their correlation, the volatility of nominal forward rates, and the correlations between forward inflation indices and nominal forward rates.

# Inflation-indexed caps

An inflation-indexed cap is a stream of inflation-indexed caplets.

An analogous definition holds for an inflation-indexed floor.

Given the set of dates $T_0, T_1, \ldots, T_M$, with $T_0 = 0$, a IICapFloor pays off, at each time $T_i$, $1, \ldots, M$,

$$N\psi_i \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa\right)\right]^+,$$

where $\kappa$ is the IICapFloor strike, $\psi_i$ are the contract year fractions for the intervals $[T_{i-1}, T_i]$, $1, \ldots, M$, $N$ is the contract nominal value, $\omega = 1$ for a cap and $\omega = -1$ for a floor.

We again set $K := 1 + \kappa$, $\mathcal{T} := \{T_1, \ldots, T_M\}$ and $\Psi := \{\psi_1, \ldots, \psi_M\}$.

# Inflation-indexed caps (cont'd)

Sticking to the second market model, from the caplet pricing formula we get:

$$\textbf{IICapFloor}(0,\mathcal{T},\Psi,K,N,\omega) = \omega N \sum_{i=1}^{M} \psi_i P_n(0,T_i)$$

$$\cdot \left[ \frac{1+\tau_i F_n(0;T_{i-1},T_i)}{1+\tau_i F_r(0;T_{i-1},T_i)} e^{D_i(0)} \Phi\left( \omega \frac{\ln \frac{1+\tau_i F_n(0;T_{i-1},T_i)}{K[1+\tau_i F_r(0;T_{i-1},T_i)]} + D_i(0) + \frac{1}{2}\mathcal{V}_i^2(0)}{\mathcal{V}_i(0)} \right) \right.$$

$$\left. -K\Phi\left( \omega \frac{\ln \frac{1+\tau_i F_n(0;T_{i-1},T_i)}{K[1+\tau_i F_r(0;T_{i-1},T_i)]} + D_i(0) - \frac{1}{2}\mathcal{V}_i^2(0)}{\mathcal{V}_i(0)} \right) \right].$$

This price depends on the volatilities of forward inflation indices and their correlations, the volatilities of nominal forward rates, and the instantaneous correlations between forward inflation indices and nominal forward rates.

# Calibration to market data

We consider an example of calibration to Euro market data as of October 7, 2004.

We calibrate the JY model and the two market models to inflation-indexed swaps.

We use the zero-coupon rates to strip the current real discount factors for the relevant maturities.

Some model parameters are fitted to ATM (nominal) caps volatilities.

The model parameters that best fit the given set of market data are found by minimizing the square absolute difference between model and market YYIIS fixed rates.

# Calibration to market data (cont'd)

To avoid over-parametrization, we introduce some constraints:

- The JY formula involves seven parameters. We reduce them to five, by finding $a_n$ and $\sigma_n$ through a previous calibration to ATM caps.
- The MM1 formula involves five parameters for each payment time from the second year onwards. We reduce them to five: $c_1, c_2, \ldots, c_5$. We set $\sigma_{r,i} = c_1/[1 + c_2(T_i - T_2)]$, $\sigma_{I,1} = c_3$ (the subsequent $\sigma_{I,i}$'s are computed accordingly), $\rho_i = c_4$ and $\rho_{I,r,i} = c_5$, for each $i > 1$.
- The MM2 formula involves five parameters for each payment time from the second year onwards. We reduce them to four: $c_1, \ldots, c_4$. We set $\rho_{I,i} = 1 - (1 - c_1)e^{-c_2 T_{i-1}}$, $\rho_{I,n,i} = c_3$ and $\sigma_{I,i} = c_4$, for each $i > 1$.

In both market models, the $\sigma_{n,i}$'s are calibrated automatically to the ATM (nominal) caps volatilities.

# Calibration to market data: results

| Maturity | Market | JY | MM1 | MM2 |
|---|---|---|---|---|
| 1 | 2.120 | 2.120 | 2.120 | 2.120 |
| 2 | 2.170 | 2.169 | 2.168 | 2.168 |
| 3 | 2.185 | 2.186 | 2.186 | 2.184 |
| 4 | 2.213 | 2.217 | 2.218 | 2.215 |
| 5 | 2.246 | 2.250 | 2.250 | 2.247 |
| 6 | 2.271 | 2.276 | 2.275 | 2.272 |
| 7 | 2.292 | 2.296 | 2.295 | 2.293 |
| 8 | 2.309 | 2.314 | 2.312 | 2.310 |
| 9 | 2.324 | 2.324 | 2.322 | 2.320 |
| 10 | 2.339 | 2.345 | 2.343 | 2.341 |
| 11 | 2.353 | 2.358 | 2.356 | 2.355 |
| 12 | 2.367 | 2.371 | 2.369 | 2.369 |
| 13 | 2.383 | 2.385 | 2.383 | 2.383 |
| 14 | 2.390 | 2.397 | 2.396 | 2.396 |
| 15 | 2.408 | 2.410 | 2.410 | 2.410 |
| 16 | 2.418 | 2.420 | 2.421 | 2.421 |
| 17 | 2.429 | 2.430 | 2.431 | 2.432 |
| 18 | 2.439 | 2.439 | 2.442 | 2.443 |
| 19 | 2.450 | 2.448 | 2.453 | 2.454 |
| 20 | 2.461 | 2.457 | 2.463 | 2.465 |

# Calibration to market data: results

The three models are equivalent in terms of calibration to market YYIIS rates. They can however imply quite different prices for zero-strike floors:

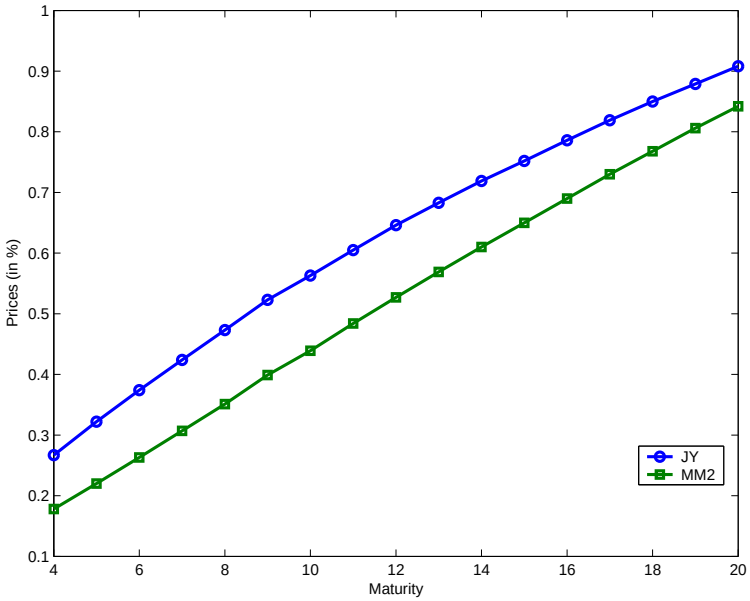



Figure 3: Comparison of zero-strike floors prices implied by the JY and second market models, for different maturities.

# Calibration to swaps and zero-strike floors

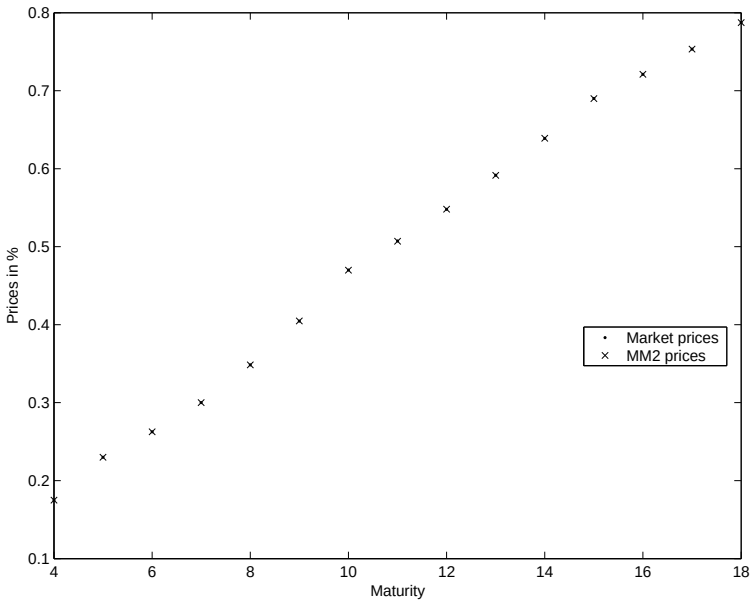



Figure 4: Zero-strike floor prices implied by the second market model, after calibration to market quotes (both swaps and floors) as of October 7, 2004.

# Including the smile: calibration to one-year caplets

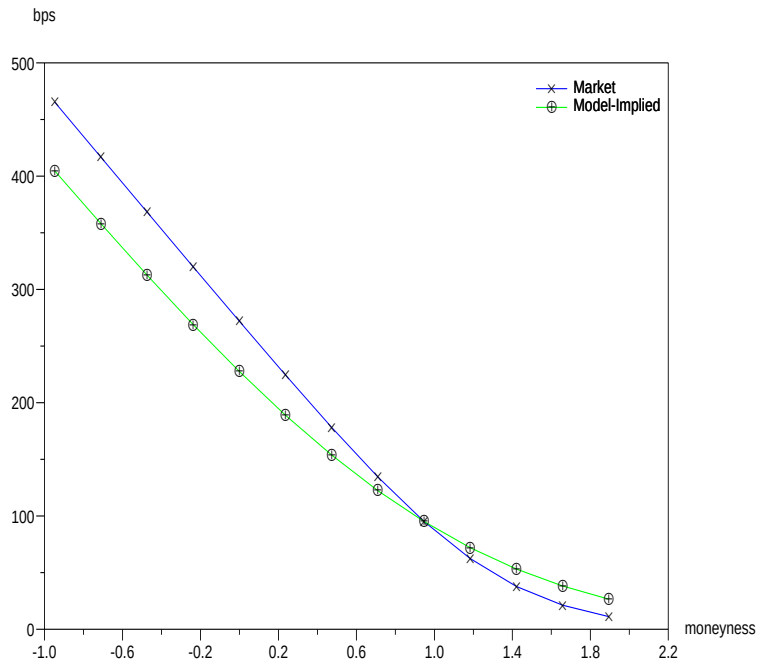



Figure 5: Market prices versus model prices with ATM implied volatility for one-year caplets. Moneyness is defined as: $K/(\mathcal{I}_1(0)/I(0) - 1)$.

# A market model with stochastic volatility

We drop the subscript $n$ and assume that, under a reference measure $\mathbb{Q}$:

- Nominal rates $F_i$ are lognormally distributed with constant volatilities;
- Forward CPI's $\mathcal{I}_i$ follow Heston-like dynamics with a common volatility process $V(t)$:

$$\begin{aligned} dF_i(t)/F_i(t) =& (\ldots)\, dt + \sigma_i^F\, dZ_i^{\mathbb{Q},F} \\ d\mathcal{I}_i(t)/\mathcal{I}_i(t) =& (\ldots)\, dt + \sigma_i^I \sqrt{V(t)}\, dZ_i^{\mathbb{Q},I} \\ dV(t) =& \alpha(\theta - V(t))\, dt + \epsilon\sqrt{V(t)}\, dW^{\mathbb{Q}}, \quad V(0) = V_0, \end{aligned}$$

  where $\sigma_i^I$, $\sigma_i^F$, $\alpha$, $\theta$, $\epsilon$ and $V_0$ are positive constants, and $2\alpha\theta > \epsilon$ to ensure positiveness of $V$.

We allow for correlations between Brownian motions $Z_i^{\mathbb{Q},F}, Z_i^{\mathbb{Q},I}, W^{\mathbb{Q}}$.

# A market model with stochastic volatility (cont'd)

We take $\mathbb{Q} = \mathbb{Q}^0$, where $\mathbb{Q}^0$ is the spot LIBOR measure corresponding to the numeraire

$$B_d(t) = P(t, \beta(t)) \prod_{l=1}^{\beta(t)} [1 + \tau_l F_l(t)], \quad \beta(t) = T_j \text{ if } T_{j-1} < t \leq T_j.$$

By definition of $B_d$ and the change-of-measure technique, we have, under $\mathbb{Q}^0$,

$$dF_i(t)/F_i(t) = \sigma_i^F \left[ - \sum_{l=\beta(t)+1}^{i} \sigma_l^F \rho_{i,l}^F \frac{\tau_l F_l(t)}{1 + \tau_l F_l(t)} dt + dZ_i^{0,F}(t) \right]$$

$$d\mathcal{I}_i(t)/\mathcal{I}_i(t) = \sqrt{V(t)}\, \sigma_i^I \left[ - \sum_{l=\beta(t)+1}^{i} \sigma_l^F \rho_{l,i}^{F,I} \frac{\tau_l F_l(t)}{1 + \tau_l F_l(t)} dt + dZ_i^{0,I}(t) \right]$$

$$dZ_i^{0,F} dZ_l^{0,F}(t) = \rho_{i,l}^F dt, \quad dZ_i^{0,I} dZ_l^{0,F}(t) = \rho_{l,i}^{F,I} dt$$

# The pricing of caplets

The price at time $t \leq T_{j-1}$ of the $j$-th caplet, is, under the measure $Q^{T_j}$,

$$\begin{aligned}\textbf{IICplt}_j(t,K) &= P(t,T_j)E_t^{T_j}\left(\frac{\mathcal{I}_j(T_j)}{\mathcal{I}_{j-1}(T_{j-1})} - K\right)^+ \\ &= P(t,T_j)\int_{-\infty}^{+\infty}(e^s - e^k)^+ q_t^j(s)\,ds\end{aligned}$$

where $k = \ln(K)$ and $q_t^j(s)ds = Q^{T_j}\left\{\ln\left[\mathcal{I}_j(T_j)/\mathcal{I}_{j-1}(T_{j-1})\right] \in [s, s+ds] | \mathcal{F}_t\right\}$.

**Remark.** Instead of having a payoff depending on a single asset $S(t)$, as it is for standard or cliquet options (paying off $[S(T_j)/S(T_{j-1}) - K]^+$ in $T_j$), here the payoff depends on the ratio between two different assets at two different times.

# The pricing of caplets (cont'd)

Following Carr and Madan (1999), we rewrite the caplet price in term of its (renormalized) Fourier transform:

$$\begin{aligned}
\textbf{IICplt}_j(t, \mathrm{e}^k) &= P(t, T_j)\frac{\mathrm{e}^{-\eta k}}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-isk}\psi_t^j(\eta, s)ds \\
&= P(t, T_j)\frac{\mathrm{e}^{-\eta k}}{\pi}\,\mathrm{Re}\int_{0}^{+\infty} \mathrm{e}^{-isk}\psi_t^j(\eta, s)ds \\
\psi_t^j(\eta, u) &= \frac{\phi_t^j(u-(\eta+1)i)}{(\eta+iu)(\eta+1+iu)}
\end{aligned}$$

where the only unknown is the conditional characteristic function $\phi_t^j(\cdot)$ of $\ln\left(\mathcal{I}_j(T_j)/\mathcal{I}_{j-1}(T_{j-1})\right)$, and where $\eta \in \mathbb{R}^+$ is used to ensure $L^2$-integrability when $k \to -\infty$.

# Derivation of the characteristic function

Our objective is now to find an explicit formula for $\phi_t^j$.

To this end, we derive the dynamics under the pricing measure $Q^{T_j}$:

$$d\mathcal{I}_j(t)/\mathcal{I}_j(t) = \sqrt{V(t)}\,\sigma_j^I\,dZ_j^I(t)$$

$$d\mathcal{I}_{j-1}(t)/\mathcal{I}_{j-1}(t) = \sqrt{V(t)}\,\sigma_{j-1}^I\left[-\frac{\tau_j F_j(t)}{1+\tau_j F_j(t)}\,\sigma_j^F\,\rho_{j,j-1}^{F,I}dt + dZ_{j-1}^I(t)\right]$$

$$dV(t) = \left[\alpha\theta - \epsilon\sqrt{V(t)}\sum_{l=\beta(t)+1}^{j}\frac{\tau_l F_l(t)}{1+\tau_l F_l(t)}\sigma_l^F\,\rho_l^{F,V} - \alpha V(t)\right]dt + \epsilon\sqrt{V(t)}\,dW(t)$$

where $dZ_l^F(t)\,dW(t) = \rho_l^{F,V}dt$, for each $l$.

# Derivation of the characteristic function (cont'd)

Setting $Y_j(t) := \ln \mathcal{I}_j(T_j)$, we recall that, by definition of characteristic function and the Markov property:

$$\phi_t^j(u) = E_t^{T_j}\left[\mathrm{e}^{iu \ln \frac{\mathcal{I}_j(T_j)}{\mathcal{I}_{j-1}(T_{j-1})}}\right] = H\left(V(t), Y_j(t), Y_{j-1}(t), F_1(t), \ldots, F_j(t)\right).$$

Applying the Feynman-Kač theorem, $H$ can then be found by solving a related PDE.

**Remark.** In the general case, due to the unpleasant presence of drift terms of type $\sqrt{V(t)}F_l(t)/(1+\tau_l F_l(t))$, there are no a priori reasons for the PDE to be explicitly solvable. In the following, we thus investigate a particular case allowing for an explicit solution.

# Derivation of the characteristic function (cont'd)

We assume that, for each $i, l = 1, \ldots, M$:

$$\rho_{i,l}^{F,I} = \rho_i^{F,V} = 0$$

We allow, however, for non-zero correlations $\rho_{j,l}^I = dZ_j^I dZ_l^I/dt$ (between different forward CPI's) and $\rho_i^{I,V} = dZ_i^I dW/dt$ (between forward CPI's and the volatility).

Setting $X_j(t) := Y_j(t) - Y_{j-1}(t)$, we then have, under $Q^{T_j}$,

$$dY_j(t) = -\frac{1}{2}V(t)(\sigma_j^I)^2\,dt + \sqrt{V(t)}\,\sigma_j^I(t)\,dZ_j^I(t)$$

$$dX_j(t) = \frac{V(t)}{2}((\sigma_{j-1}^I)^2 - (\sigma_j^I)^2)\,dt + \sqrt{V(t)}(\sigma_j^I\,dZ_j^I(t) - \sigma_{j-1}^I\,dZ_{j-1}^I(t))$$

$$dV(t) = [\alpha\theta - \alpha V(t)]\;dt + \epsilon\sqrt{V(t)}\,dW(t)$$

# Derivation of the characteristic function (cont'd)

To make $\phi_t^j$ explicit, we write

$$\phi_t^j(u) = E_t^{T_j}\left[\mathrm{e}^{iu\left(Y_j(T_j)-Y_{j-1}(T_{j-1})\right)}\right] = E_t^{T_j}\left[\mathrm{e}^{-iuY_{j-1}(T_{j-1})}E_{T_{j-1}}^{T_j}\left(\mathrm{e}^{iuY_j(T_j)}\right)\right]$$

Noting that $E_{T_{j-1}}^{T_j}\left(\mathrm{e}^{iuY_j(T_j)}\right)$ is the characteristic function of $\ln \mathcal{I}_j(T_j)$ conditional on $\mathcal{F}_{T_{j-1}}$, solving a Heston-like PDE, we have that

$$E_{T_{j-1}}^{T_j}\left[\mathrm{e}^{iuY_j(T_j)}\right] = \exp\left\{A_Y(\bar{\tau}_j, u) + B_Y(\bar{\tau}_j, u)V(T_{j-1}) + iuY_j(T_{j-1})\right\}$$

where $\bar{\tau}_j := T_j - T_{j-1}$ and $A_Y$ and $B_Y$ are deterministic complex functions.

Consequently,

$$\phi_t^j(u) = \mathrm{e}^{A_Y(\bar{\tau}_j,u)}E_t^{T_j}\left[\mathrm{e}^{iuX_j(T_{j-1})+B_Y(\bar{\tau}_j,u)V(T_{j-1})}\right]$$

# Derivation of the characteristic function (cont'd)

The last conditional expectation is nothing but the characteristic function of the couple $(X_j(T_{j-1}), V(T_{j-1}))$ evaluated at point $(u, -iB_Y(\bar{\tau}_j, u))$.

By again solving a PDE of Heston's type with suitable boundary conditions, we obtain

$$\phi_t^j(u) = \exp\left\{A_Y(\bar{\tau}_j, u) + A_X(T_{j-1} - t, u) + B_X(T_{j-1} - t, u)V(t) + iuX_j(t)\right\}$$

where $A_X$ and $B_X$ are other deterministic complex functions.

The II caplet price is finally calculated by numerical integration:

$$\textbf{IICplt}_j(t, \mathrm{e}^k) = P(t, T_j)\frac{\mathrm{e}^{-\eta k}}{\pi}\,\mathrm{Re}\int_0^{+\infty} \mathrm{e}^{-isk}\frac{\phi_t^j(s - (\eta+1)i)}{(\eta + is)(\eta + 1 + is)}\,ds$$

# Derivation of the characteristic function (cont'd)

**The coefficients $A_Y$ and $B_Y$:**

$$B_Y(s,u) = \frac{\gamma - b}{2a}\left[\frac{1-\mathrm{e}^{\gamma s}}{1-\frac{b-\gamma}{b+\gamma}\mathrm{e}^{\gamma s}}\right]$$

$$A_Y(s,u) = \frac{\alpha\theta(\gamma-b)}{2a}s - \frac{\alpha\theta}{a}\ln\left[\frac{1-\frac{b-\gamma}{b+\gamma}\mathrm{e}^{\gamma s}}{1-\frac{b-\gamma}{b+\gamma}}\right]$$

where

$$a := \epsilon^2/2, \quad c := -iu(\sigma_j^I)^2/2 - (\sigma_j^I)^2u^2/2,$$

$$b := iu\sigma_j^I\epsilon\rho_j^{I,V} - \alpha, \quad \gamma := \sqrt{b^2-4ac}.$$

# Derivation of the characteristic function (cont'd)

**The coefficients $A_X$ and $B_X$:**

$$B_X(\tau,u) = B_Y(\tau_j,u) + \frac{\bar{\gamma}-\bar{b}-2\bar{a}B_Y(\tau_j,u)}{2\bar{a}}\left[\frac{1-\mathrm{e}^{\bar{\gamma}\tau}}{1-\frac{2\bar{a}B_Y(\tau_j,u)+\bar{b}-\bar{\gamma}}{2\bar{a}B_Y(\tau_j,u)+\bar{b}+\bar{\gamma}}\mathrm{e}^{\bar{\gamma}\tau}}\right]$$

$$A_X(\tau,u) = \frac{\alpha\theta(\bar{\gamma}-\bar{b})}{2\bar{a}}\tau - \frac{\alpha\theta}{\bar{a}}\ln\left[\frac{1-\frac{2\bar{a}B_Y(\tau_j,u)+\bar{b}-\bar{\gamma}}{2\bar{a}B_Y(\tau_j,u)+\bar{b}+\bar{\gamma}}\mathrm{e}^{\bar{\gamma}\tau}}{1-\frac{2\bar{a}B_Y(\tau_j,u)+\bar{b}-\bar{\gamma}}{2\bar{a}B_Y(\tau_j,u)+\bar{b}+\bar{\gamma}}}\right]$$

where

$$\bar{a} := \epsilon^2/2, \qquad \bar{b} := iu\epsilon(\sigma_j^I\rho_j^{I,V} - \sigma_{j-1}^I\rho_{j-1}^{I,V}) - \alpha$$

$$\bar{c} := iu\big((\sigma_{j-1}^I)^2 - (\sigma_j^I)^2\big)/2 - \big((\sigma_{j-1}^I)^2 + (\sigma_j^I)^2 - 2\sigma_j^I\sigma_{j-1}^I\rho_{j,j-1}^I\big)u^2/2$$

$$\bar{\gamma} := \sqrt{\bar{b}^2 - 4\bar{a}\bar{c}}$$

## Solving the general case with non-zero correlations

As in a LIBOR market model, we can deal with forward-rates-dependent terms in the drifts by resorting to a freezing technique.

The drift terms that involve forward rates are

$$D_l(t) := \sqrt{V(t)} \frac{F_l(t)}{1 + \tau_l F_l(t)}$$

and depend on the volatility, too.

A first way to freeze these terms consists in setting

$$D_l(t) \approx D_l(0),$$

thus changing the asymptotic volatility value from $\theta$ to

$$\tilde{\theta} := \theta - \frac{\epsilon}{\alpha} \sum_{l=1}^{j} D_l(0) \tau_l \sigma_l^F \rho_l^{F,V}$$

# Solving the general case with non-zero correlations (cont'd)

The first freezing leads to the following (approximated) SDEs for $X_j$ and $V$:

$$
\begin{aligned}
dX_j(t) \approx & \left[\frac{V(t)}{2}\left((\sigma_{j-1}^I)^2 - (\sigma_j^I)^2\right) + D_j(0)\tau_j\sigma_{j-1}^I\sigma_j^F\rho_{j,j-1}^{F,I}\right] dt \\
& + \sqrt{V(t)}[\sigma_j^I\, dZ_j^I(t) - \sigma_{j-1}^I\, dZ_{j-1}^I(t)] \\
dV(t) \approx & \alpha(\tilde{\theta} - V(t))\, dt + \epsilon\sqrt{V(t)}\, dW(t).
\end{aligned}
$$

The dynamics of $X_j$ here differs from the previous one for a constant drift term.

Such a term, however, is innocuous and the relevant characteristic functions can still be calculated explicitly.

# Solving the general case with non-zero correlations (cont'd)

A second possibility for a tractable approximation is to set

$$D_l(t) \approx \frac{F_l(t)}{1+\tau_l F_l(t)} \frac{V(t)}{\sqrt{V(t)}} \approx \frac{F_l(0)}{1+\tau_l F_l(0)} \frac{V(t)}{\sqrt{V(0)}} = D_l(0) \frac{V(t)}{V(0)},$$

where the freezing is done with the purpose of producing a linear term in $V(t)$. This leads to the following (approximated) SDEs for $X_j$ and $V$:

$$\begin{aligned} dX_j(t) \approx & V(t) \left[ \frac{1}{2} \left( (\sigma_{j-1}^I)^2 - (\sigma_j^I)^2 \right) + \frac{D_j(0)}{V(0)} \tau_j \sigma_{j-1}^I \sigma_j^F \rho_{j,j-1}^{F,I} \right] dt \\ & + \sqrt{V(t)} [\sigma_j^I \, dZ_j^I(t) - \sigma_{j-1}^I \, dZ_{j-1}^I(t)] \\ dV(t) \approx & \bar{\alpha}(\bar{\theta} - V(t)) \, dt + \epsilon \sqrt{V(t)} \, dW(t), \end{aligned}$$

# Solving the general case with non-zero correlations (cont'd)

where

$$\bar{\alpha} := \alpha + \frac{\epsilon}{V(0)} \sum_{l=1}^{j} D_l(0) \tau_l \sigma_l^F \rho_l^{F,V}$$

$$\bar{\theta} := \alpha\theta / \bar{\alpha}$$

Also in this second case, the relevant characteristic functions can be calculated explicitly.

**Remark.** For the approximated processes $V(t)$ to be meaningful, we must require $\tilde{\theta} > 0$ and $\bar{\alpha} > 0$. Moreover, for the origin to be inaccessible, conditions $2\tilde{\theta}\alpha > \epsilon$ and $2\bar{\theta}\bar{\alpha} > \epsilon$ must be imposed, with the latter that is automatically satisfied since $2\theta\alpha > \epsilon$ by assumption.

# Solving the general case with non-zero correlations (cont'd)

To test the goodness of the above approximations, we should perform a Monte Carlo simulation and compare the analytical caplet prices coming from the approximations with the corresponding Monte Carlo price windows calculated numerically.

This procedure, however, is rather cumbersome, since it also requires the joint simulation of all forward rates.

A much quicker test can be conducted by simply comparing the caplet prices implied by the two approximations.

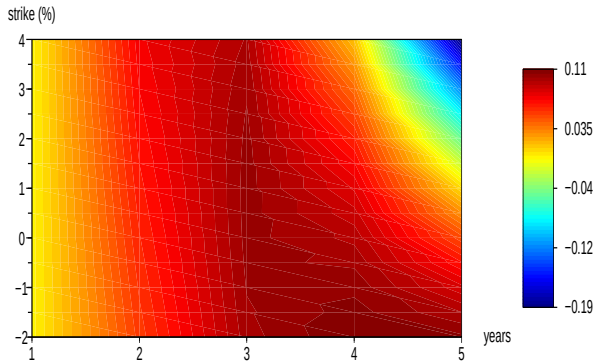
The two approximations, moreover, can be compared with the exact price obtained in the "zero-correlation case".

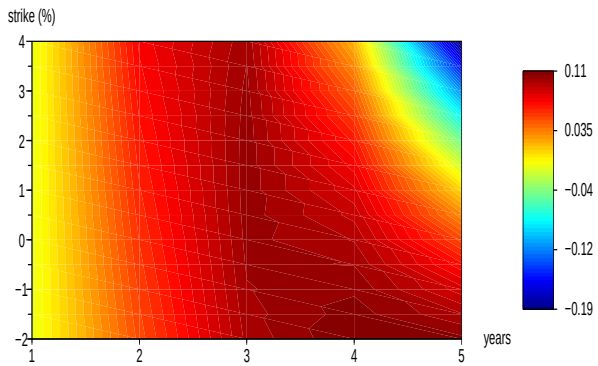
# Solving the general case with non-zero correlations (cont'd)

We test the goodness of the above approximations on EUR data as of October 7, 2004.

We set: $\rho_i^{F,V} = \rho_{i,l}^{F,I} = \rho_i^{I,V} = -0.2$,
$\rho_{i,i-1}^{I} = 1 - 1.5\mathrm{e}^{-0.08(i-2)}$,
$\alpha = 0.2$, $\theta = 0.001$, $V(0) = 0.001$,
$\epsilon = 0.01$ and
$\sigma_i^I = 1 - 0.05(i-1)$, for $i, l = 1, \ldots, 5$.

We plot the percentage differences between "exact" and "freezing-based" caplets prices.
The first freezing case is shown on top.



# Calibration to a matrix of II caps/floors

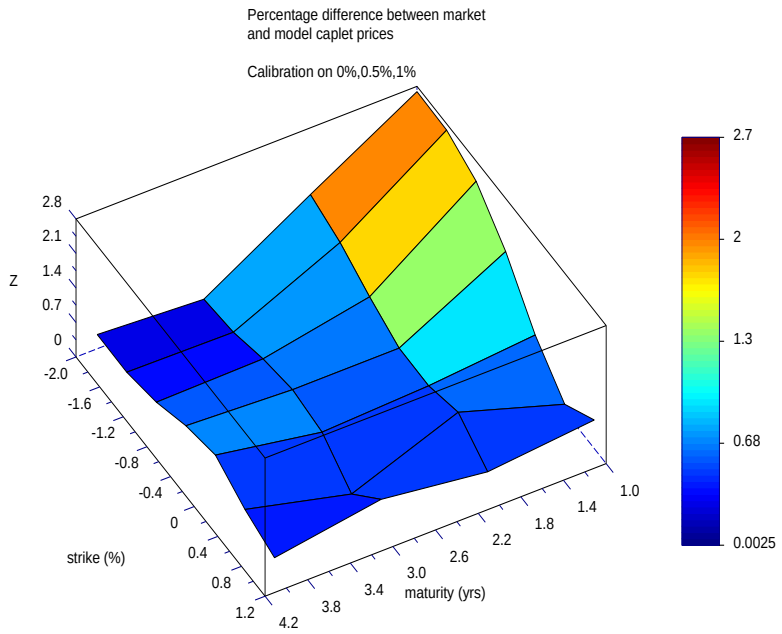



Figure 6: Absolute percentage differences between calibrated prices and market prices (market quotes as of October 7, 2004).

# Conclusions

The classical pricing of inflation-indexed derivatives requires the modelling of both nominal and real rates and of the reference consumer price index.

The foreign-currency analogy allows one to view real rates as the rates in a foreign economy and to treat the CPI as an exchange rate.

Assuming a Gaussian distribution for both nominal and real rates, as in the JY model, we have provided explicit formulas for II swaps and caps.

We have also described two alternative market models leading to analytical formulas with easier-to-estimate input parameters.

The second market model has been extended by allowing for a stochastic volatility as in Heston (1993).

We have finally presented examples of calibration to II market data.