SWAPTION SMILE AND CMS ADJUSTMENT

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Stylized facts

- Information on the implied volatilities of swap rates is provided by the market both directly through the quoted swaption smiles and indirectly through prices of CMS swaps.

- Unfortunately, not every swaption in the ATM matrix has also away-from-the-money quotes. Similarly, CMS swap spreads are only available for few swap maturities and CMS tenors.

- The following questions, therefore, arise naturally:
  - What is the value of implied volatilities for strikes outside (and in particular larger than) the quoted ones?
  - How can one strip CMS convexity adjustments from the quoted CMS swap spreads?

Stylized facts (cont'd)

- To address such issues in a sound and rigorous way, one should resort to a dynamical interest-rate model, as for instance the market models of Piterbarg (2003) or Henry-Labordere (2006).

- However, this approach may be too time-consuming and does not necessarily grant the precision required for a good estimate of the missing volatility values.

- One can thus prefer to resort to a less elegant solution, which however answers the above questions in a rapid and consistent way.

- In this paper, we propose an empirical, yet consistent, procedure that serves this purpose, and make use of the SABR functional form, which has become a standard in the market for modelling swaption volatilities.

Outline of the talk

- We start by defining a CMS convexity adjustment and show how to value it consistently with the swaption smile by using static replication arguments.
- We will see that CMS convexity adjustments are affected by the swaption smiles and, conversely, that the asymptotic values of implied volatilities are affected by the prices of CMS swaps.
- We then introduce the SABR functional form to model (implied) swaption volatilities and calculate the associated adjustments.
- We finally consider an example of joint calibration to swaption volatilities and CMS swap spreads.
- We conclude by using the extrapolated swaption volatilities to calculate a smile-consistent CMS option price.

The convexity adjustment

Let us fix a maturity T_a and a set of times $T_{a+1}, \ldots, T_b$, with associated year fractions all equal to $\tau > 0$. The associated forward swap rate at time t is defined by

$$S_{a,b}(t) = \frac{P(t, T_a) - P(t, T_b)}{\tau \sum_{j=a+1}^{b} P(t, T_j)},$$

where $P(t, T)$ denotes the time-t discount factor for maturity T.

The convexity adjustment for the swap rate $S_{a,b}(T_a)$ is defined by

$$\mathbf{CA}(S_{a,b}; \delta) := E^{T_a+\delta}(S_{a,b}(T_a)) - S_{a,b}(0),$$

where δ is the accrual period of the swap rate, and E^T denotes expectation under the T-forward measure.

The convexity adjustment (cont'd)

The quantity $\mathbf{CA}(S_{a,b};\delta)$ depends on the entire evolution of the yield curve, which makes it impossible, in general, to calculate it exactly in closed form.

Two are the ad-hoc approximations one may resort to:

a) Static replication of a CMS-swaplet's payoff by an infinite combination of cash-settled swaptions:

$$S = f(0)SG_{a,b}(S) + \int_0^{+\infty} [f''(x)x + 2f'(x)](S-x)^+ G_{a,b}(S)\,dx,$$

where

$$G_{a,b}(S) := \sum_{j=1}^{b-a} \frac{\tau}{(1+\tau S)^j}, \quad f(S) := \frac{1}{G_{a,b}(S)}$$

The convexity adjustment (cont'd)

b) Approximation of the Radon-Nykodim derivative associated to the relevant forward and swap measures, in terms of a function of the swap rate $S_{a,b}(T_a)$ itself (implicitly using physically-settled swaptions):

$$E^{T_a+\delta}(S_{a,b}(T_a)) = \frac{\sum_{j=a+1}^{b} P(0,T_j)}{P(0,T_a+\delta)} E^{a,b}\left(\frac{S_{a,b}(T_a)P(T_a,T_a+\delta)}{\sum_{j=a+1}^{b} P(T_a,T_j)}\right)$$

$$\approx E^{a,b}\left(\frac{S_{a,b}(T_a)\bar{f}(S_{a,b}(T_a);\delta)}{\bar{f}(S_{a,b}(0);\delta)}\right)$$

where

$$\bar{f}(S;\delta) := \frac{f(S)}{(1+\tau S)^{\delta}} = \frac{1}{(1+\tau S)^{\delta}\sum_{j=1}^{b-a}\frac{\tau}{(1+\tau S)^{j}}}$$

The convexity adjustment (cont'd)

Remarkably, under the assumption that the implied volatilities of cash-settled and physically-settled swaptions are equal when the underlying swap rate is the same, the two approximations yield the same result:

$$E^{T_a+\delta}[S_{a,b}(T_a)] \approx \frac{1}{\bar{f}(S_{a,b}(0))}\Bigg[\bar{f}(0)S_{a,b}(0) + \int_0^{+\infty}[\bar{f}''(x)x + 2\bar{f}'(x)]c_{a,b}(x)\,dx\Bigg]$$

For the sake of tractability, one typically uses, in practice, a further approximation, which is based on a first-order Taylor expansion, see also Hagan (2003):

$$\bar{f}(S) \approx \bar{f}(S_{a,b}(0)) + \bar{f}'(S_{a,b}(0))[S - S_{a,b}(0)]$$

The convexity adjustment (cont'd)

We thus obtain:

$$\mathbf{CA}(S_{a,b};\delta) \approx S_{a,b}(0)\,\theta(\delta)\left(\frac{E^{a,b}\big(S_{a,b}^2(T_a)\big)}{S_{a,b}^2(0)} - 1\right)$$

where

$$\theta(\delta) := 1 - \frac{\tau S_{a,b}(0)}{1+\tau S_{a,b}(0)}\left(\delta + \frac{b-a}{(1+\tau S_{a,b}(0))^{b-a}-1}\right).$$

Assuming lognormal-type dynamics for $S_{a,b}$ under the swap measure $Q^{a,b}$, we get the classical (Black-like) adjustment:

$$\mathbf{CA}^{\text{Black}}(S_{a,b};\delta) \approx S_{a,b}(0)\,\theta(\delta)\left(e^{(\sigma_{a,b}^{\text{ATM}})^2\,T_a} - 1\right)$$

where $\sigma_{a,b}^{\text{ATM}}$ denotes the ATM implied volatility for $S_{a,b}$.

The convexity adjustment (cont'd)

In presence of a market smile, the adjustment is necessarily more involved.

In fact, denoting by $\sigma^{\mathsf{M}}_{a,b}(K)$ the market implied volatility of a (physically-settled) swaption with strike K, standard replication arguments imply that

$$E^{a,b}\big(S^2_{a,b}(T_a)\big) = 2\int_0^\infty \mathsf{BI}\big(K, S_{a,b}(0), v^{\mathsf{M}}_{a,b}(K)\big)\, dK$$

where

$$\mathsf{BI}(K,S,v) := S\Phi\left(\frac{\ln(S/K)+v^2/2}{v}\right) - K\Phi\left(\frac{\ln(S/K)-v^2/2}{v}\right),$$

$$v^{\mathsf{M}}_{a,b}(K) := \sigma^{\mathsf{M}}_{a,b}(K)\sqrt{T_a}$$

A CMS convexity adjustment is thus affected by the whole smile of the associated swap rate.

The convexity adjustment (cont'd)

However, volatility quotes are provided up to some $\bar{K}$, so that only

$$\int_0^{\bar{K}} \mathsf{BI}(K, S_{a,b}(0), v_{a,b}^{\mathsf{M}}(K))\, dK$$

can be inferred from market swaption data.

Since the residual term (integral from $\bar{K}$ to infinity) is not negligible in general, a CMS adjustment can not be fully determined by the quoted smile.

This motivates a joint calibration to swaptions and CMS swap spreads, which serves a twofold purpose:

- Inferring the asymptotic behavior of implied volatilities;
- Stripping CMS adjustments.

This purpose will be accomplished by modelling implied volatilities with the SABR functional form.

The SABR functional form

Hagan *et al.* (2002) propose a stochastic volatility model for the evolution of the forward price of a given asset under the asset's canonical measure.

In case of a swap rate, one assumes that $S_{a,b}(t)$ evolves under the associated swap measure $Q^{a,b}$ according to

$$\begin{aligned} dS_{a,b}(t) &= V(t) S_{a,b}(t)^{\beta}\, dZ^{a,b}(t), \\ dV(t) &= \epsilon V(t)\, dW^{a,b}(t), \\ V(0) &= \alpha, \end{aligned}$$

where $Z^{a,b}$ and $W^{a,b}$ are $Q^{a,b}$-standard Brownian motions with

$$dZ^{a,b}(t)\, dW^{a,b}(t) = \rho\, dt,$$

and where $\beta \in (0,1]$, ϵ and α are positive constants and $\rho \in [-1,1]$.

The SABR functional form (cont'd)

Using singular perturbation techniques, Hagan *et al.* (2002) derive the following formula for implied volatilities:

$$\sigma^{\mathsf{imp}}(K, S_{a,b}(0)) \approx \frac{\alpha}{(S_{a,b}(0)K)^{\frac{1-\beta}{2}}\left[1+\frac{(1-\beta)^2}{24}\ln^2\left(\frac{S_{a,b}(0)}{K}\right)+\frac{(1-\beta)^4}{1920}\ln^4\left(\frac{S_{a,b}(0)}{K}\right)\right]}$$

$$\cdot\,\frac{z}{x(z)}\left\{1+\left[\frac{(1-\beta)^2\alpha^2}{24(S_{a,b}(0)K)^{1-\beta}}+\frac{\rho\beta\epsilon\alpha}{4(S_{a,b}(0)K)^{\frac{1-\beta}{2}}}+\epsilon^2\frac{2-3\rho^2}{24}\right]T_a\right\},$$

where $z := \frac{\epsilon}{\alpha}(S_{a,b}(0)K)^{\frac{1-\beta}{2}}\ln\left(\frac{S_{a,b}(0)}{K}\right)$ and $x(z) := \ln\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}$.

Even though this is only an approximation, it is market practice to consider it as exact and to use it as a functional form mapping strikes into implied volatilities.

The SABR functional form (cont'd)

Under the SABR functional form, the previous formula for the CMS convexity adjustment becomes

$$\mathbf{CA}^{\mathsf{SABR}}(S_{a,b};\delta) = S_{a,b}(0)\,\theta(\delta) \cdot \left(\frac{2}{S_{a,b}^2(0)}\int_0^\infty \mathsf{BI}\big(K, S_{a,b}(0), v^{\mathsf{imp}}(K, S_{a,b}(0))\big)\,dK - 1\right),$$

where $v^{\mathsf{imp}}(K, S_{a,b}(0)) := \sigma^{\mathsf{imp}}(K, S_{a,b}(0))\sqrt{T_a}$.

The integral in the RHS is finite since the SABR implied volatility satisfies Lee's (2004) condition (bounded for large strikes by $\sqrt{\ln K}$). The only exception is the case $\beta = 1$, where we have:

$$\lim_{K\to+\infty} \frac{\sigma^{imp}(K, S_{a,b}(0))}{\sqrt{\ln K}} = \lim_{K\to+\infty} \frac{\epsilon\sqrt{\ln K}}{\ln\left(\frac{\epsilon}{2\alpha}\ln K\right)} = +\infty.$$

The market data

Swaption volatilities are quoted in the market for different strikes K as a difference $\Delta\sigma^{\mathsf{M}}_{a,b}$ with respect to the ATM level

$$\Delta\sigma^{\mathsf{M}}_{a,b}(\Delta K) := \sigma^{\mathsf{M}}_{a,b}(K^{\mathsf{ATM}} + \Delta K) - \sigma^{\mathsf{ATM}}_{a,b}$$

usually for $\Delta K \in \{\pm 200, \pm 100, \pm 50, \pm 25, 0\}$ basis points.

Market implied volatilities do not allow, by themselves, to uniquely identify the four parameters of the SABR functional form.

In fact, different combinations of β and ρ can accommodate the market volatilities quotes for given maturity and tenor.

An "implied" value for the β parameter can only be inferred as soon as we include suitable non "plain vanilla" instruments, such as CMS swaps, in our data set.

The market data (cont'd)

The market also quotes the spread $X_{n,c}$ over LIBOR that sets to zero the value of a CMS swap paying the c-year swap rate on dates T'_i, $i = 1, \ldots, n$.

Denoting by $S'_{i,c}$ the c-year (forward) swap rate setting at $T'_i - \delta$, the spread is explicitly given in terms of CMS convexity adjustments as:

$$X_{n,c} = \frac{\sum_{i=1}^{n} \left(S'_{i,c}(0) + \mathbf{CA}(S'_{i,c}; \delta) \right) P(0, T'_i)}{\sum_{i=1}^{n} P(0, T'_i)} - \frac{1 - P(0, T'_n)}{\delta \sum_{i=1}^{n} P(0, T'_i)},$$

where all the accrual periods are equal to $\delta = 3m$.

In our calibration example, we will use this equation to infer convexity adjustments $\mathbf{CA}(S'_{n,c}; \delta)$ from market spreads $X_{n,c}$.

Remark. Sometimes, prices of CMS caps or floors can also be found in the market. However, for calibration purposes, we will limit ourselves to CMS swaps, since they are usually more liquid than CMS options.

The market data (cont'd)

		Strike							
Expiry	**Tenor**	-200	-100	-50	-25	25	50	100	200
1y	10y	7.68%	2.33%	0.80%	0.31%	-0.15%	-0.17%	0.10%	1.16%
2y	10y	7.90%	2.50%	0.90%	0.40%	-0.20%	-0.30%	-0.20%	0.60%
5y	10y	6.54%	2.30%	0.93%	0.41%	-0.30%	-0.51%	-0.68%	-0.39%
10y	10y	5.24%	1.92%	0.81%	0.37%	-0.29%	-0.52%	-0.80%	-0.83%
20y	10y	5.19%	1.92%	0.81%	0.37%	-0.30%	-0.53%	-0.82%	-0.88%
30y	10y	5.20%	1.92%	0.81%	0.37%	-0.31%	-0.55%	-0.87%	-1.01%
1y	20y	7.53%	2.49%	0.92%	0.38%	-0.21%	-0.28%	-0.09%	0.91%
5y	20y	6.43%	2.29%	0.92%	0.41%	-0.30%	-0.50%	-0.65%	-0.31%
10y	20y	5.36%	1.96%	0.82%	0.37%	-0.29%	-0.51%	-0.76%	-0.71%
20y	20y	5.32%	1.96%	0.83%	0.37%	-0.30%	-0.53%	-0.79%	-0.78%
30y	20y	5.44%	2.00%	0.84%	0.38%	-0.31%	-0.55%	-0.84%	-0.88%
1y	30y	7.63%	2.58%	0.96%	0.40%	-0.23%	-0.31%	-0.12%	0.90%
5y	30y	6.52%	2.31%	0.93%	0.41%	-0.30%	-0.49%	-0.61%	-0.20%
10y	30y	5.51%	2.01%	0.83%	0.38%	-0.29%	-0.51%	-0.75%	-0.66%
20y	30y	5.45%	2.01%	0.84%	0.38%	-0.31%	-0.54%	-0.80%	-0.78%
30y	30y	5.53%	2.04%	0.86%	0.39%	-0.32%	-0.56%	-0.86%	-0.89%

EUR market volatility smiles as of 3 February 2006. Strikes are expressed as absolute differences in basis points $w.r.t$ the ATM values.

The market data (cont'd)

	Tenor		
Expiry	10y	20y	30y
1y	15.50%	13.40%	12.80%
5y	15.00%	14.00%	13.50%
10y	13.80%	12.90%	12.50%
20y	12.50%	11.50%	11.30%
30y	12.30%	11.60%	11.60%

Eur market ATM volatilities as of 3 February 2006.

	Tenor		
Maturity	10y	20y	30y
5y	64.9	84.8	87.1
10y	60.8	76.2	78.6
15y	56.0	68.4	71.6
20y	52.6	64.3	68.9
30y	51.9	65.4	73.6

Eur market CMS swap spreads (in basis points) as of 3 February 2006.

Calibration example

Our example of calibration is based on Euro data as of 3 February 2006.

Our calibration is performed by minimizing the squared percentage difference between model quantities (swaption volatilities and CMS swap spreads) and the corresponding market ones.

Implied volatilities are given by the SABR functional form. We assume that:

- Each swap rate is associated with different parameters α, ϵ and ρ;
- The parameter β is equal across different maturities and tenors;

Remark. We are not considering a swap model in a strict sense, but simply assuming a (static) functional form for swaption volatilities.

Calibration example (cont'd)

We denote by:

- $\mathcal{S}$ the set of expiry-tenor pairs (a, b) for the quoted swaptions;
- $\mathcal{X}$ the set of expiry-tenor pairs (n, c) for the quoted CMS swap spreads.

The joint calibration to swaption smiles and CMS convexity adjustments is carried out through the following two-stage procedure:

1. Set the β parameter at an initial guess.
2. For each expiry-tenor pair $(a, b) \in \mathcal{S}$:
   - Select initial values $\alpha^0 = \alpha^0_{a,b}$, $\epsilon^0 = \epsilon^0_{a,b}$ and $\rho^0 = \rho^0_{a,b}$.
   - Calibrate parameters $\alpha = \alpha_{a,b}$, $\epsilon = \epsilon_{a,b}$ and $\rho = \rho_{a,b}$ to the corresponding swaption smile, obtaining $\alpha_{a,b}(\beta)$, $\epsilon_{a,b}(\beta)$ and $\rho_{a,b}(\beta)$.
   - Calculate the CMS adjustment with parameters β, $\alpha_{a,b}(\beta)$, $\epsilon_{a,b}(\beta)$, $\rho_{a,b}(\beta)$.

Calibration example (cont'd)

3. For each market pair $(n, c) \in \mathcal{X}$, with $c = b - a$:
   - Calculate the corresponding CMS swap spread using the adjustments $\mathbf{CA}(S'_{i,c}; \delta)$ calculated in the previous step.
   - Denote the obtained spread by $X_{n,c}(\beta)$.

4. Iterate over β, repeating steps 2 to 4, until the CMS swap spreads $X_{n,c}(\beta)$, for all market pairs $(n, c) \in \mathcal{X}$, are as close as possible to the corresponding market quotes.

Market data can be accommodated in a quite satisfactory way. Calibration errors are within typical bid-ask spreads.

Calibration results

	Swaption								
Strike	5x10	5x20	5x30	10x10	10x20	10x30	20x10	20x20	20x30
-200	1.1	0.2	0.3	0.2	0.1	0.4	0.8	0.6	0.7
-100	0.5	0.2	0.2	0.3	0.1	0.2	0.6	0.1	0.7
-50	0.8	0.2	0.4	0.4	0.5	0.3	0.4	0.9	0.2
-25	0.4	0.4	0.5	0.4	0.2	0.5	0.4	0.1	0.2
0	0.1	0.0	0.2	0.1	0.1	0.1	0.0	0.1	0.3
25	0.3	0.1	0.3	0.1	0.1	0.2	0.3	0.2	0.4
50	0.1	0.1	0.3	0.3	0.3	0.1	0.3	0.6	0.5
100	0.2	0.2	0.4	0.9	0.8	0.8	0.5	0.2	0.1
200	0.4	0.1	0.4	0.0	0.0	0.2	0.5	0.3	0.3

Absolute differences (in bp) between market and SABR implied volatilities.

Calibration results (cont'd)

	Tenor		
Maturity	10y	20y	30y
5y	0.8	1.5	2.4
10y	1.7	2.9	4.4
15y	1.8	3.8	6.2
20y	1.3	4.0	7.5
30y	2.1	1.3	2.4

Absolute differences (in bp) between market CMS swap spreads and those induced by the SABR functional form.

Calibration of the β parameter

Though β can assume any value between zero and one, in practice we have to bound it to achieve a successful calibration.

In fact, values of β approaching one lead to divergent values for convexity adjustments (for $\beta = 1$ the correction is infinite).

	$\boldsymbol{\beta}$						
Maturity	0.2	0.3	0.4	0.5	0.6	0.7	0.8
5y	9.6	10.5	10.6	10.7	10.7	10.8	11.0
10y	19.7	19.9	20.1	20.4	20.8	21.5	23.0
20y	25.4	32.5	33.8	35.7	39.3	49.4	155.4
30y	32.5	43.9	46.7	51.7	62.9	107.6	1326.8

CMS swap spreads $X_{n,10}(\beta)$ for different maturities n, after calibration, with fixed β, to whole swaption smile.

The pricing of CMS options

Once the joint calibration to CMS swaps and swaption smiles is completed, CMS caplet prices can be calculated as follows:

$$E^T[(S_{a,b}(T_a) - K)^+]$$
$$= c_{a,b}(K) + \theta(T - T_a)\left[\frac{2}{S_{a,b}(0)}\int_K^\infty c_{a,b}(x)\,dx - \frac{S_{a,b}(0) - K}{S_{a,b}(0)}c_{a,b}(K)\right]$$

where $c_{a,b}(x) := \mathsf{Bl}(x, S_{a,b}(0), v^{\mathsf{imp}}(x, S_{a,b}(0)))$.

On February 3rd, 2006, the only available quote in the Euro market was that of a ten-year CMS cap with maturity 12 years and strike 4.75%. The market bid-ask was $350 - 365$ bp. The formula above leads to a price of 361.1.

Conclusions

We have proposed a very simple and robust procedure for extrapolating both swaption volatilities for non quoted strikes and CMS convexity corrections for relevant expiries and tenors.

Our procedure is based on parameterizing swaption volatilities by means of the SABR functional form, which leads to an explicit formula for CMS convexity adjustments.

We have also provided an example of calibration to market data, assuming a unique β parameter for all swap rates. This can be generalized, for instance, by considering different values of β for swap rates with different tenors.

Using the calibrated SABR parameters, we can then price CMS options, which are very sensitive to the swaption smile, without resorting to a fully-consistent interest-rate model.