Consistent Pricing and Hedging of FX Derivatives

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Stylized facts

- In the FX market three volatility quotes are available for a number of maturities: 0Δ straddle, Risk Reversal and Vega-Weighted Butterfly.

- From these quotes, one can immediately obtain the volatilities for the 25Δ call and the 25Δ put (besides the ATM one).

- The market makes use of an empirical procedure to construct the whole smile for a given maturity.

- Volatility quotes are then provided in terms of the option's Δ, for ranges from the 5Δ put to the 5Δ call.

- Interpolation schemes are usually employed for non-canonical maturities.

Outline of the talk

- We review the market empirical procedure to construct the smile surface for a given currency.

- We derive closed-form formulas so as to render the construction procedure more explicit.

- We then test the robustness (in a static sense) of the constructed smile:
  - Changing consistently the initial pairs (strike, vol) leads to the same smile;
  - The same procedure applied to European-style claims is consistent with static-replication results.

- We finally prove that the market empirical procedure defines a hedging strategy that is replicating and self-financing at first order in the maturity of the considered option.

The market quotes

The ATM volatility quoted in the FX option market is that of a 0Δ straddle:

$$\sigma_{\mathrm{ATM}} = \sigma_{0\Delta s}$$

The 25Δ Risk Reversal (RR) is a structure where one buys a 25Δ call and sells a 25Δ put. It is quoted as the difference between the two implied volatilities, $\sigma_{25\Delta c}$ and $\sigma_{25\Delta p}$:

$$\sigma_{\mathrm{RR}} = \sigma_{25\Delta c} - \sigma_{25\Delta p}$$

The 25Δ Vega-Weighted Butterfly (VWB) is built up by selling an ATM straddle and buying a 25Δ strangle. Denoting by σ_{VWB} the butterfly's price in volatility terms, we have:

$$\sigma_{\mathrm{VWB}} = \frac{\sigma_{25\Delta c} + \sigma_{25\Delta p}}{2} - \sigma_{\mathrm{ATM}}$$

The market quotes (cont'd)

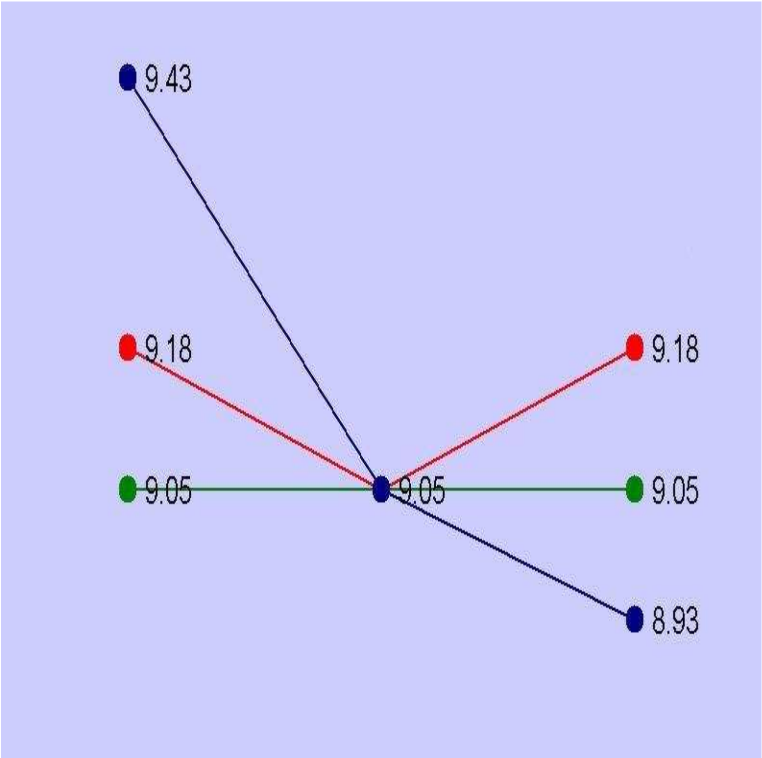
Solving the previous linear system:

$$\sigma_{50\Delta c} = \sigma_{50\Delta p} = \sigma_{\text{ATM}}$$

$$\sigma_{25\Delta c} = \sigma_{\text{ATM}} + \sigma_{\text{VWB}} + \frac{1}{2}\sigma_{\text{RR}}$$

$$\sigma_{25\Delta p} = \sigma_{\text{ATM}} + \sigma_{\text{VWB}} - \frac{1}{2}\sigma_{\text{RR}}$$

$$\begin{array}{lcl} \sigma_{\text{ATM}} = 9.05\% & & \sigma_{50\Delta c} = 8.93\% \\ \sigma_{\text{RR}} = -0.50\% & \Rightarrow & \sigma_{25\Delta c} = 9.05\% \\ \sigma_{\text{VWB}} = 0.13\% & & \sigma_{25\Delta p} = 9.43\% \end{array}$$



The corresponding strikes

The ATM strike

The ATM volatility is that of a 0Δ straddle (Δ call = - Δ put). Therefore,

$$e^{-r^f T}\Phi\left(\frac{\ln\frac{S_0}{K_{\text{ATM}}}+(r^d-r^f+\frac{1}{2}\sigma^2_{\text{ATM}})T}{\sigma_{\text{ATM}}\sqrt{T}}\right)$$

$$= e^{-r^f T}\Phi\left(-\frac{\ln\frac{S_0}{K_{\text{ATM}}}+(r^d-r^f+\frac{1}{2}\sigma^2_{\text{ATM}})T}{\sigma_{\text{ATM}}\sqrt{T}}\right)$$

$$\Leftrightarrow \ln\frac{S_0}{K_{\text{ATM}}}+(r^d-r^f+\frac{1}{2}\sigma^2_{\text{ATM}})T = 0$$

$$\Leftrightarrow K_{ATM} = S_0\, e^{(r^d-r^f+\frac{1}{2}\sigma^2_{ATM})T}$$

The corresponding strikes (cont'd)

The 25Δ strikes

By definition, for a 25Δ put we must have that

$$- e^{-r^f T}\Phi\left(- \frac{\ln \frac{S_0}{K_{25\Delta p}} + (r^d - r^f + \frac{1}{2}\sigma^2_{25\Delta p})T}{\sigma_{25\Delta p}\sqrt{T}}\right) = -0.25$$

$$\Leftrightarrow K_{25\Delta p} = S_0\, e^{-\alpha\sigma_{25\Delta p}\sqrt{T}+(r^d-r^f+\frac{1}{2}\sigma^2_{25\Delta p})T}$$

where $\alpha := -\Phi^{-1}(\frac{1}{4}e^{r^f T})$.

Similarly, for a 25Δ call, one gets:

$$K_{25\Delta c} = S_0\, e^{\alpha\sigma_{25\Delta c}\sqrt{T}+(r^d-r^f+\frac{1}{2}\sigma^2_{25\Delta c})T}$$

The empirical market procedure

Consider a European call option with time to maturity $\tau := T - t$ and strike K, whose Black and Scholes (BS) price, at time t, is

$$\begin{aligned} C^{\mathrm{BS}}(t;K) =& S_t\, e^{-r^f\tau}\Phi\left(\frac{\ln\frac{S_t}{K} + (r^d - r^f + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\ & - K e^{-r^d\tau}\Phi\left(\frac{\ln\frac{S_t}{K} + (r^d - r^f - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \end{aligned}$$

Under the BS model the call payoff can be replicated by a dynamic Delta-hedging strategy.

In real financial markets, however, volatility is stochastic and traders hedge by constructing Vega-neutral portfolios. In the FX case, portfolios can also be constructed so as to match partial derivatives up to the second order.

The empirical market procedure (cont'd)

We now show how to derive such a hedging portfolio in the case of the above call with maturity T and strike K.

The portfolio is made of three European calls with maturity T and strikes $K_1 = K_{25\Delta p}$, $K_2 = K_{\mathrm{ATM}}$ and $K_3 = K_{25\Delta c}$.

Problem. Assuming a Delta-hedged position, find time-t weights $x_1(t;K)$, $x_2(t;K)$ and $x_3(t;K)$ such that the resulting portfolio hedges the price variations of the call up to the second order in S and σ.

N.B. Remember that, in a BS world, Vega neutral portfolios of plain-vanilla options (with the same maturity) are also Gamma neutral. In fact

$$\frac{1}{2}\sigma\frac{\partial C^{\mathrm{BS}}}{\partial\sigma} = \frac{\tau}{2}\sigma^2 S_t^2\frac{\partial^2 C^{\mathrm{BS}}}{\partial S^2} \quad \text{(Vega-Gamma relationship)}$$

The empirical market procedure (cont'd)

The weights $x_1(t;K)$, $x_2(t;K)$ and $x_3(t;K)$ can be found by imposing that the "replicating" portfolio has the same Vega, dVegadVol and dVegadSpot as the call with strike K, namely

$$\text{same Vega} \Rightarrow \quad \frac{\partial C^{\text{BS}}}{\partial \sigma}(t;K) = \sum_{i=1}^{3} x_i(t;K) \frac{\partial C^{\text{BS}}}{\partial \sigma}(t;K_i)$$

$$\text{same dVegadVol} \Rightarrow \quad \frac{\partial^2 C^{\text{BS}}}{\partial^2 \sigma}(t;K) = \sum_{i=1}^{3} x_i(t;K) \frac{\partial^2 C^{\text{BS}}}{\partial^2 \sigma}(t;K_i)$$

$$\text{same dVegadSpot} \Rightarrow \quad \frac{\partial^2 C^{\text{BS}}}{\partial \sigma \partial S_t}(t;K) = \sum_{i=1}^{3} x_i(t;K) \frac{\partial^2 C^{\text{BS}}}{\partial \sigma \partial S_t}(t;K_i)$$

The empirical market procedure (cont'd)

We denote by $\mathcal{V}(t;K)$ the time-t Vega of a European option with (maturity T and) strike K,

$$\mathcal{V}(t;K) = \frac{\partial C^{\mathrm{BS}}}{\partial \sigma}(t;K) = S_t\, e^{-r^f \tau} \sqrt{\tau} \varphi(d_1(t;K))$$

$$d_1(t;K) = \frac{\ln \frac{S_t}{K} + (r^d - r^f + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad \varphi(x) = \Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

and calculate the second order derivatives:

$$\frac{\partial^2 C^{\mathrm{BS}}}{\partial^2 \sigma}(t;K) = \frac{\mathcal{V}(t;K)}{\sigma} d_1(t;K) d_2(t;K)$$

$$\frac{\partial^2 C^{\mathrm{BS}}}{\partial \sigma \partial S_t}(t;K) = -\frac{\mathcal{V}(t;K)}{S_t \sigma \sqrt{\tau}} d_2(t;K), \quad d_2(t;K) = d_1(t;K) - \sigma\sqrt{\tau}$$

The empirical market procedure (cont'd)

After straightforward algebra, we obtain that the above system (of three equations in the three unknowns x_1, x_2 and x_3) admits always a unique solution given by

$$x_1(t;K) = \frac{\mathcal{V}(t;K)}{\mathcal{V}(t;K_1)} \frac{\ln\frac{K_2}{K}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_1}}$$

$$x_2(t;K) = \frac{\mathcal{V}(t;K)}{\mathcal{V}(t;K_2)} \frac{\ln\frac{K}{K_1}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_2}}$$

$$x_3(t;K) = \frac{\mathcal{V}(t;K)}{\mathcal{V}(t;K_3)} \frac{\ln\frac{K}{K_1}\ln\frac{K}{K_2}}{\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}}$$

In particular, if $K = K_j$ then $x_i(t;K) = 1$ for $i = j$ and zero otherwise.

The empirical market procedure (cont'd)

A "smile-consistent" price for the call with strike K is obtained by adding to the BS price the cost of implementing the above hedging strategy at prevailing market prices. In formulas $(t = 0)$:

$$C(K) = C^{\text{BS}}(K) + \sum_{i=1}^{3} x_i(K)[C^{\text{MKT}}(K_i) - C^{\text{BS}}(K_i)]$$

When $K = K_j$, $C(K_j) = C^{\text{MKT}}(K_j)$, since $x_i(K_j) = \delta_{i,j}$.

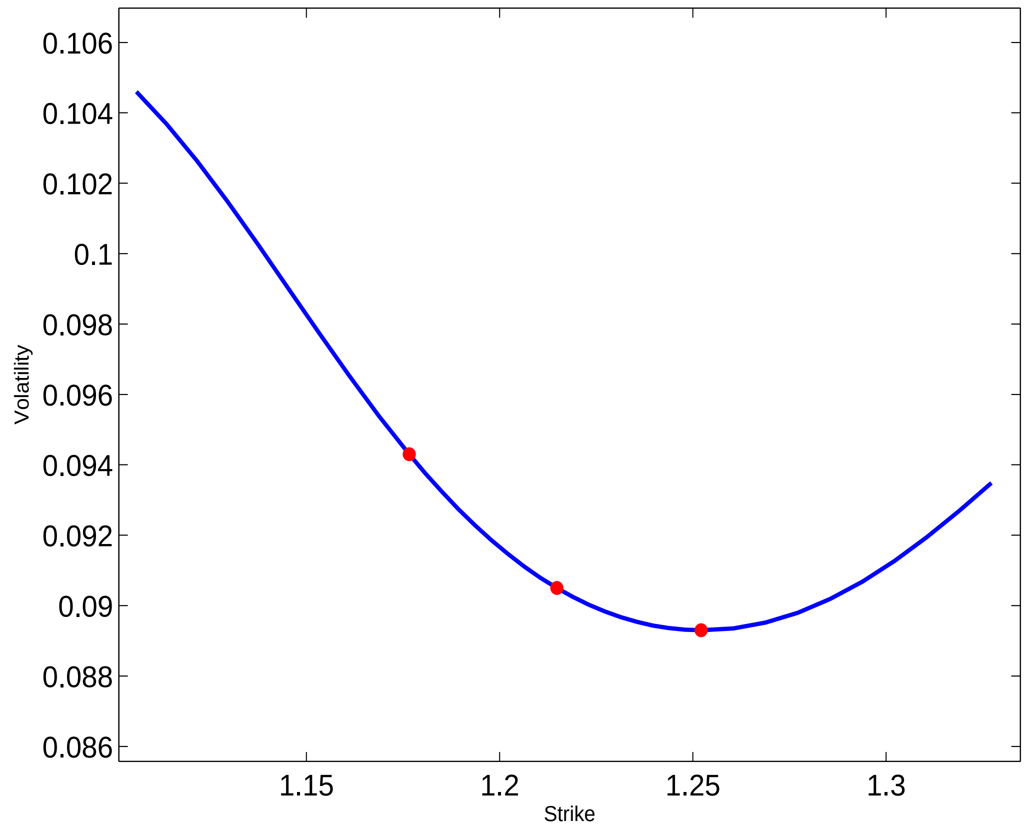
This formula, therefore, defines a rule for interpolating (extrapolating) prices from the three quotes $C^{\text{MKT}}(K_1)$, $C^{\text{MKT}}(K_2)$ and $C^{\text{MKT}}(K_3)$.

A market implied volatility curve can then be constructed by inverting $C(K)$, for each considered K, through the BS formula.

The empirical market procedure (cont'd)

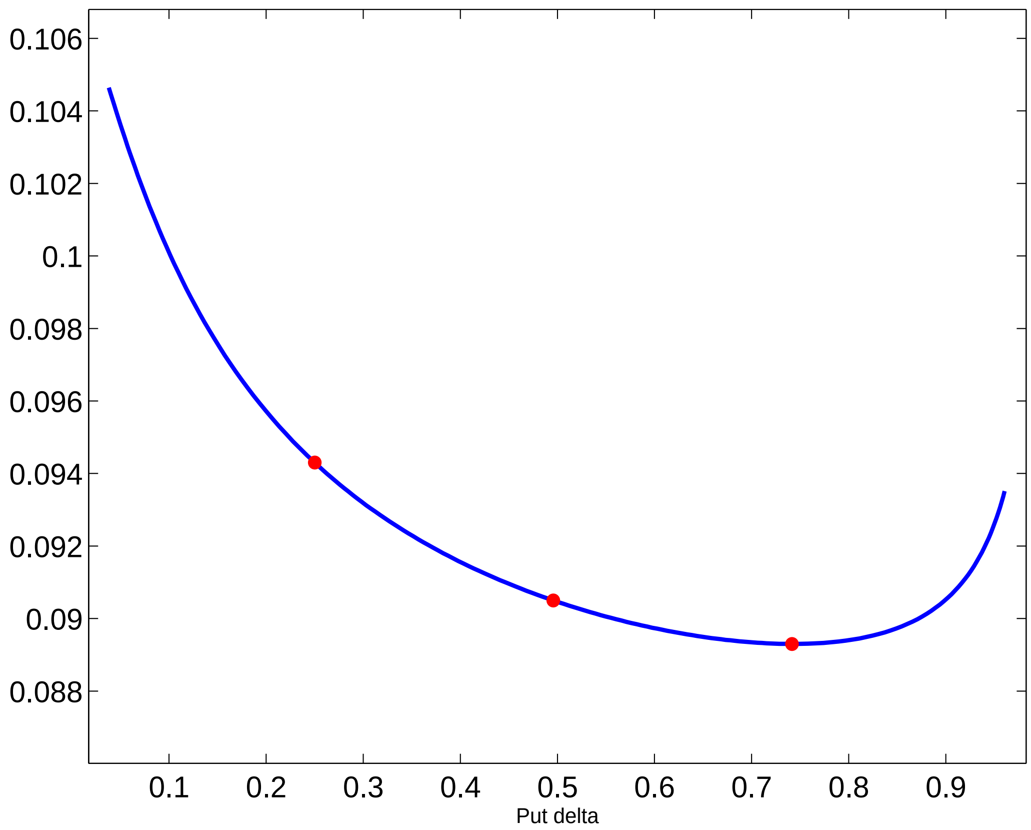
EURUSD data as of 1 July 2005

$T = 3m \ (= 94/365y)$
$S_0 = 1.205$
$\sigma_{\text{ATM}} = 9.05\%$
$\sigma_{\text{RR}} = -0.50\%$
$\sigma_{\text{VWB}} = 0.13\%$
$\Rightarrow$
$\sigma_{50\Delta c} = 8.93\%$
$\sigma_{25\Delta c} = 9.05\%$
$\sigma_{25\Delta p} = 9.43\%$
$K_{\text{ATM}} = 1.2148$
$K_{25\Delta c} = 1.1767$
$K_{25\Delta p} = 1.2521$



The empirical market procedure (cont'd)

Implied volatilities plotted against put Deltas



No-arbitrage conditions

The option price $C(K)$, as a function of the strike K, satisfies the following no-arbitrage conditions:

i) $C \in C^2((0, +\infty))$;

ii) $\lim_{K \to 0^+} C(K) = S_0 \, e^{-r^f T}$ and $\lim_{K \to +\infty} C(K) = 0$;

iii) $\lim_{K \to 0^+} \frac{dC}{dK}(K) = -e^{-r^d T}$ and $\lim_{K \to +\infty} K \frac{dC}{dK}(K) = 0$.

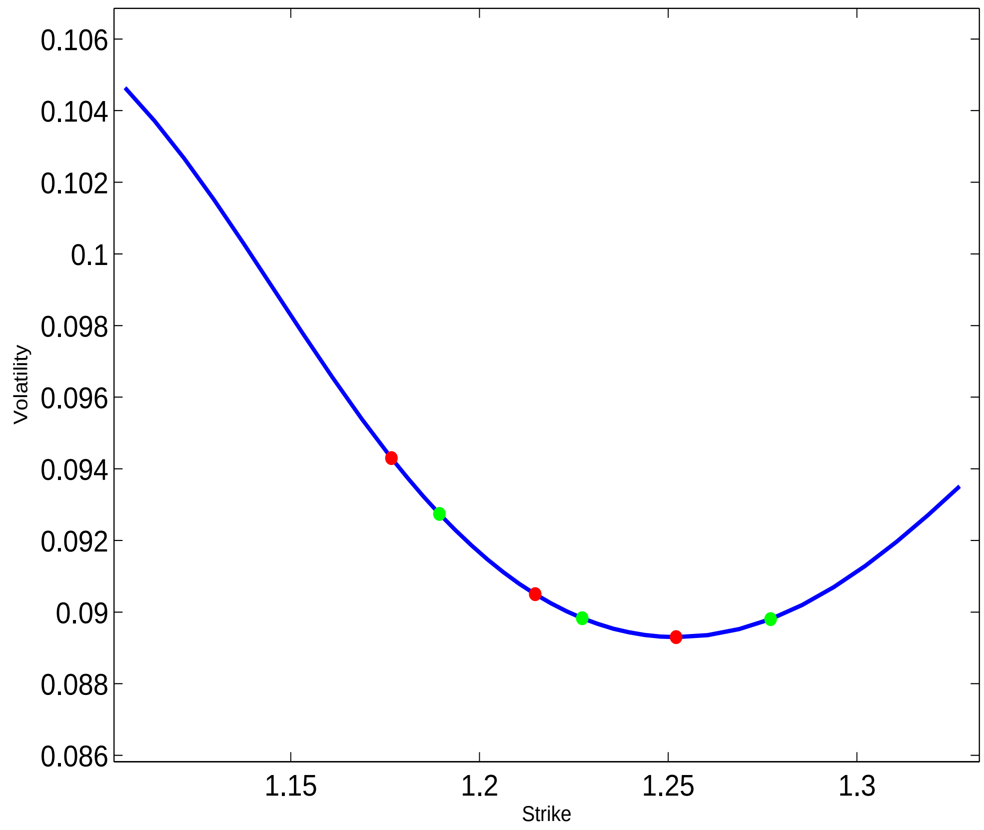
Properties **ii)** and **iii)** (obviously satisfied by C^{BS}) hold, since, for each i, both $x_i(K)$ and $dx_i(K)/dK$ go to zero for $K \to 0^+$ or $K \to +\infty$.

The option price $C(K)$ should also be a convex function of the strike K, *i.e.* $\frac{d^2C}{dK^2}(K) > 0$ for each $K > 0$. This property, which is not true in general, holds however for typical market parameters.

A first consistency result

Smile invariance

Any other triplet
$(H_1, \sigma(H_1))$
$(H_2, \sigma(H_2))$
$(H_3, \sigma(H_3))$
in the smile generated by
$(K_1, \sigma(K_1))$
$(K_2, \sigma(K_2))$
$(K_3, \sigma(K_3))$
lead to the same implied volatility curve



A second consistency result

Let us consider a European-style payoff $h(S_T)$ at time T, and denote by V^h its price at time 0. By static replication arguments, we have

$$V^h = e^{-r^d T} h(0) + S_0\, e^{-r^f T} h'(0) + \int_0^{+\infty} h''(x) C(x)\, dx$$

Analogously to the call with strike K, let us then define a new (smile consistent) price as

$$\overline{V}^h = V^{h,\mathsf{BS}} + \sum_{i=1}^{3} x_i^h [C(K_i) - C^{\mathsf{BS}}(K_i)]$$

where weights x_i^h are found, as before, by matching Vega, dVegadVol and dVegadSpot in a BS world.

Proposition. The two prices V^h and $\overline{V}^h$ coincide.

Consistent pricing of a quanto option

A quanto option is a derivative paying out at maturity T the amount

$$[\omega(S_T - X)]^+ \quad \text{in foreign currency}$$

which is equivalent to

$$[\omega(S_T - X)]^+ S_T \quad \text{in domestic currency}$$

where $\omega = 1$ for a call and $\omega = -1$ for a put.

Standard arguments on static replication imply that quanto call and put prices can be written as

$$\mathsf{QCall}(X) = 2\int_X^{+\infty} C(K)\,dK + XC(X)$$

$$\mathsf{QPut}(X) = XP(X) - 2\int_0^X P(K)\,dK$$

Consistent pricing of a quanto option (cont'd)

A numerical example showing that quanto option prices based on static replication and on hedging arguments coincide.

Strike	1.1750		1.2050		1.2350	
Expiry	3M	1Y	3M	1Y	3M	1Y
Hedging arguments						
Call	4.8917%	8.7404%	2.8409%	6.7434%	1.4301%	5.0545%
Put	0.7935%	1.8740%	1.7173%	2.8031%	3.2812%	4.0401%
Static replication (500 steps)						
Call	4.8963%	8.7275%	2.8460%	6.7381%	1.4325%	5.0548%
Abs Diff.	0.005%	-0.013%	0.005%	-0.005%	0.002%	0.000%
Put	0.7877%	1.8690%	1.7145%	2.8005%	3.2750%	4.0396%
Abs Diff.	-0.006%	-0.005%	-0.003%	-0.003%	-0.006%	0.000%
Static replication (3000 steps)						
Call	4.8916%	8.7383%	2.8433%	6.7434%	1.4311%	5.0570%
Abs Diff.	0.000%	-0.002%	0.002%	0.000%	0.001%	0.002%
Put	0.7885%	1.8711%	1.7164%	2.8034%	3.2785%	4.0433%
Abs Diff.	-0.005%	-0.003%	-0.001%	0.000%	-0.003%	0.003%

Robustness of the empirical market procedure

Our pricing formula can also be formally justified in dynamical terms.

In fact, we can prove that, if the basic European options are valued with a unique (stochastic) implied volatility, the value changes of the hedging portfolio locally track those of the call with strike K.

To this end, we consider a generic time t and assume Ito-like dynamics for the (implied) volatility $\sigma = \sigma_t$.

We also assume a Delta-hedged position and that the strikes K_i are those derived at the initial time.

By Ito's lemma, we have the following:

$$dC_t^{\mathrm{BS}}(K) - \sum_{i=1}^{3} x_i(t;K)\, dC_t^{\mathrm{BS}}(K_i) = \left[\frac{\partial C_t^{\mathrm{BS}}(K)}{\partial t} - \sum_{i=1}^{3} x_i(t;K) \frac{\partial C_t^{\mathrm{BS}}(K_i)}{\partial t}\right] dt + \ldots$$

Robustness of the empirical market procedure (cont'd)

$$\dots + \left[\frac{\partial C_t^{\mathsf{BS}}(K)}{\partial \sigma} - \sum_{i=1}^{3} x_i(t;K)\frac{\partial C_t^{\mathsf{BS}}(K_i)}{\partial \sigma}\right] d\sigma_t$$

$$+ \frac{1}{2}\left[\frac{\partial^2 C_t^{\mathsf{BS}}(K)}{\partial S^2} - \sum_{i=1}^{3} x_i(t;K)\frac{\partial^2 C_t^{\mathsf{BS}}(K_i)}{\partial S^2}\right] (dS_t)^2$$

$$+ \frac{1}{2}\left[\frac{\partial^2 C_t^{\mathsf{BS}}(K)}{\partial \sigma^2} - \sum_{i=1}^{3} x_i(t;K)\frac{\partial^2 C_t^{\mathsf{BS}}(K_i)}{\partial \sigma^2}\right] (d\sigma_t)^2$$

$$+ \left[\frac{\partial^2 C_t^{\mathsf{BS}}(K)}{\partial S \partial \sigma} - \sum_{i=1}^{3} x_i(t;K)\frac{\partial^2 C_t^{\mathsf{BS}}(K_i)}{\partial S \partial \sigma}\right] dS_t d\sigma_t$$

The second, fourth and fifth terms are zero by definition of the weights x_i, whereas the third is zero due to the Vega-Gamma relationship.

Robustness of the empirical market procedure (cont'd)

For the same reason, we also have

$$\frac{\partial C_t^{\mathrm{BS}}(K)}{\partial t} - \sum_{i=1}^{3} x_i(t;K)\frac{\partial C_t^{\mathrm{BS}}(K_i)}{\partial t} = r^d\left[C_t^{\mathrm{BS}}(K) - \sum_{i=1}^{3} x_i(t;K) C_t^{\mathrm{BS}}(K_i)\right]$$

so that

$$dC_t^{\mathrm{BS}}(K) - \sum_{i=1}^{3} x_i(t;K)\, dC_t^{\mathrm{BS}}(K_i) = r^d\left[C_t^{\mathrm{BS}}(K) - \sum_{i=1}^{3} x_i(t;K) C_t^{\mathrm{BS}}(K_i)\right] dt$$

The portfolio made of a long position in the call with strike K and three short positions in $x_i(t;K)$ calls with strike K_i is *locally riskless* at time t.

Robustness of the empirical market procedure (cont'd)

Defining

$$\xi(t) = \frac{C_t^{\mathsf{BS}}(K) - \sum_{i=1}^{3} x_i(t;K) C_t^{\mathsf{BS}}(K_i)}{B(t)}$$

where $B(t) = \exp(r^d t)$ is the domestic bank account value at time t, the strategy

$$(x_1(t;K), x_2(t;K), x_3(t;K), \xi(t))$$

replicates the BS price of the option in that, for each t,

$$C_t^{\mathsf{BS}}(K) = \sum_{i=1}^{3} x_i(t;K) C_t^{\mathsf{BS}}(K_i) + \xi(t) B(t)$$

and is also self-financing since

$$dC^{\mathsf{BS}}(t;K) = \sum_{i=1}^{3} x_i(t;K)\, dC^{\mathsf{BS}}(t;K_i) + \xi(t)\, dB(t)$$

Robustness of the empirical market procedure (cont'd)

Under actual market prices, the hedging portfolio replicates the option payoff when the maturity is relatively short. In fact, for a small $T = \tau$,

$$(S_T - K)^+ - C^{\mathrm{BS}}(K) - \sum_{i=1}^{3} x_i(K)[(S_T - K_i)^+ - C^{\mathrm{BS}}(K_i)]$$

$$\approx r^d \left[C^{\mathrm{BS}}(K) - \sum_{i=1}^{3} x_i(K) C^{\mathrm{BS}}(K_i) \right] \tau$$

Applying our price definition for $C(K)$, we immediately get

$$(S_T - K)^+ - C(K) - \sum_{i=1}^{3} x_i(K)[(S_T - K_i)^+ - C^{\mathrm{MKT}}(K_i)]$$

$$\approx r^d \left[C(K) - \sum_{i=1}^{3} x_i(K) C^{\mathrm{MKT}}(K_i) \right] \tau$$

An approximation for implied volatilities

At first order in σ, one has

$$C(K) = C^{\mathsf{BS}}(K) + \sum_{i=1}^{3} x_i(K)[C^{\mathsf{MKT}}(K_i) - C^{\mathsf{BS}}(K_i)]$$

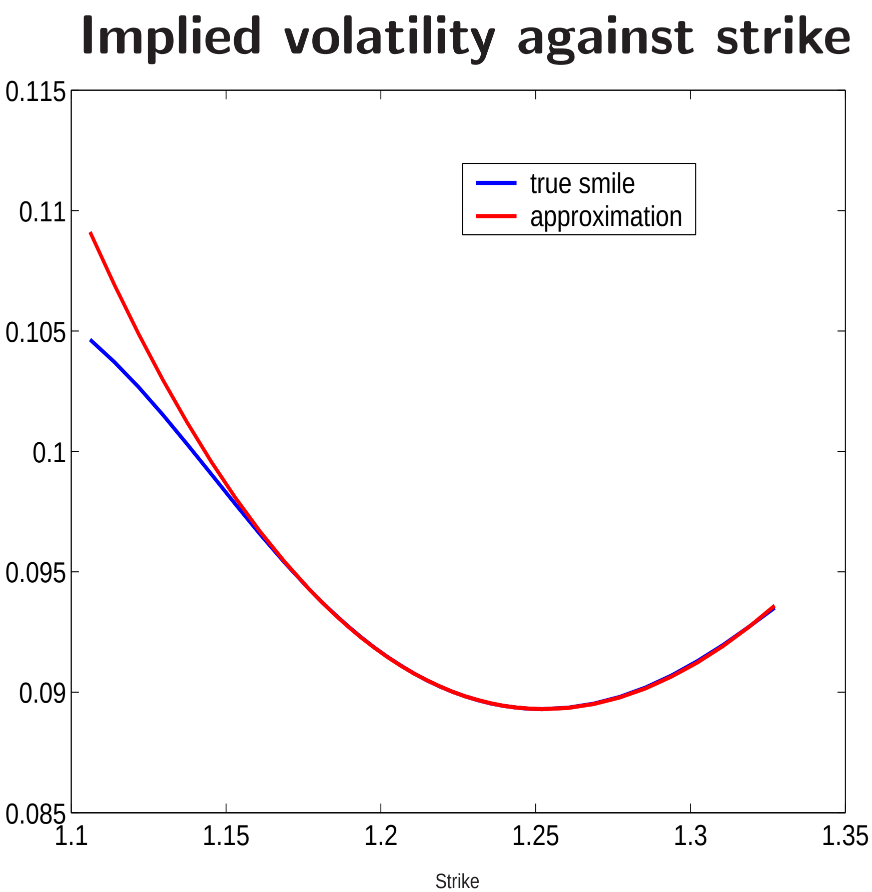
$$\approx C^{\mathsf{BS}}(K) + \sum_{i=1}^{3} x_i(K)\mathcal{V}(K_i)[\sigma(K_i) - \sigma]$$

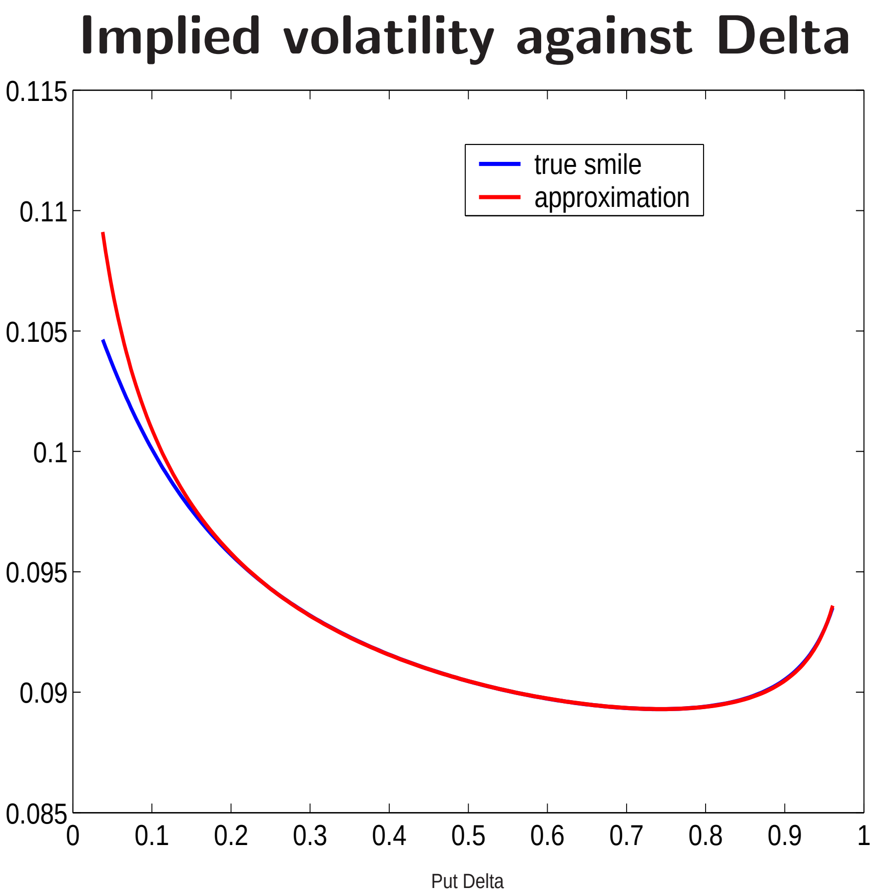
$$\approx C^{\mathsf{BS}}(K) + \mathcal{V}(K)\left[\sum_{i=1}^{3} y_i(K)\sigma(K_i) - \sigma\right]$$

where coefficients $y_i(K)$ do not depend on σ and sum up to one.

$$\Rightarrow \quad \sigma(K) \approx \frac{\ln\frac{K_2}{K}\,\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\,\ln\frac{K_3}{K_1}}\sigma_{25\Delta p} + \frac{\ln\frac{K}{K_1}\,\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\,\ln\frac{K_3}{K_2}}\sigma_{\mathrm{ATM}} + \frac{\ln\frac{K}{K_1}\,\ln\frac{K}{K_2}}{\ln\frac{K_3}{K_1}\,\ln\frac{K_3}{K_2}}\sigma_{25\Delta c}$$

An approximation for implied volatilities (cont'd)





Conclusions

We have described a market procedure to construct implied volatility curves in the FX market.

We have seen that the smile construction procedure leads to a pricing formula for any European-style contingent claim.

We have then proven consistency results based on static replication and on hedging arguments.

The smile construction procedure and the related pricing formula can be applied in any market where three volatility quotes are available for a given maturity.

The valuation of exotic claims is, in general, a more complex issue to deal with.