No-Arbitrage Conditions for a Finite Options System

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Abstract

In this document we derive necessary and sufficient conditions for a finite system of option prices (with the same maturity) to be arbitrage free. We start by considering the particular case of three market quotes (European calls), which is typical of the FX options market. We then derive stricter conditions by allowing also the trading in forward contracts and in the underlying asset. We finally hint at the general case of an arbitrary, but finite, number of given option prices.

Notation and Conventions

S_t: the underlying asset price at time t.
r: the (constant) domestic instantaneous risk-free rate.
q: the (constant) foreign instantaneous risk-free rate.
T: a fixed maturity.
$C(X)$: the market price of a European call option with maturity T and strike X.
$P(X)$: the market price of a European put option with maturity T and strike X.

We call *positive* the real numbers strictly larger than zero. Any number larger than or equal to zero is then called nonnegative.

1.1 The Case of Three European Call Prices

In a given options market, we consider a maturity T and three strikes K_1, K_2 and K_3, $K_1 < K_2 < K_3$. We denote by $C(K_1)$, $C(K_2)$ and $C(K_3)$ the related option prices.

In the following, we derive necessary and sufficient conditions for the three prices to form an arbitrage-free system.[1]

Proposition 1.1. *The three option prices $C(K_1)$, $C(K_2)$ and $C(K_3)$ form an arbitrage-free system if and only if*

$$\begin{cases} C(K_3) > 0 \\ C(K_2) > C(K_3) \\ (K_3 - K_2)C(K_1) - (K_3 - K_1)C(K_2) + (K_2 - K_1)C(K_3) > 0 \end{cases} \tag{1}$$

[1] These conditions can also be found in Cox and Rubinstein (1985).

Proof. The option prices $C(K_1)$, $C(K_2)$ and $C(K_3)$ form an arbitrage-free system if and only if every linear combination of the related calls yielding a nonnegative payoff that is positive with positive probability, has a positive price.

Since, for any (real) combinators α_1, α_2 and α_3,

$$\sum_{i=1}^{3} \alpha_i (x - K_i)^+ = \begin{cases} 0 & x \leq K_1 \\ \alpha_1(x - K_1) & K_1 < x \leq K_2 \\ \alpha_1(x - K_1) + \alpha_2(x - K_2) & K_2 < x \leq K_3 \\ (\alpha_1 + \alpha_2 + \alpha_3)x - \alpha_1 K_1 - \alpha_2 K_2 - \alpha_3 K_3 & x > K_3 \end{cases}$$

we then have

$$\sum_{i=1}^{3} \alpha_i (S_T - K_i)^+ \geq 0 \Leftrightarrow \begin{cases} \alpha_1 \geq 0 \\ \alpha_1(K_3 - K_1) + \alpha_2(K_3 - K_2) \geq 0 \\ \alpha_1 + \alpha_2 + \alpha_3 \geq 0 \end{cases}$$

Setting $\bar{\alpha} := (\alpha_1, \alpha_2, \alpha_3)$ and

$$\mathcal{A} := \{\bar{\alpha} \neq (0,0,0) : \alpha_1 \geq 0,\ \alpha_1(K_3 - K_1) + \alpha_2(K_3 - K_2) \geq 0,\ \alpha_1 + \alpha_2 + \alpha_3 \geq 0\}$$

we have therefore an arbitrage-free system if and only if

$$\sum_{i=1}^{3} \alpha_i C(K_i) > 0 \text{ for each } \bar{\alpha} \in \mathcal{A} \tag{2}$$

Defining the new variables vector $\bar{\beta} = (\beta_1, \beta_2, \beta_3)$ as

$$\begin{cases} \beta_1 = \alpha_1 \\ \beta_2 = \alpha_1(K_3 - K_1) + \alpha_2(K_3 - K_2) \\ \beta_3 = \alpha_1 + \alpha_2 + \alpha_3 \end{cases}$$

condition (2) is equivalent to

$$\beta_1 \Big(C(K_1) - \frac{K_3 - K_1}{K_3 - K_2} C(K_2) + \frac{K_2 - K_1}{K_3 - K_2} C(K_3) \Big) + \beta_2 \frac{C(K_2) - C(K_3)}{K_3 - K_2} + \beta_3 C(K_3) > 0 \tag{3}$$

for each $\bar{\beta} \in \mathcal{B} := \{\bar{\beta} \neq (0,0,0) : \beta_1 \geq 0,\ \beta_2 \geq 0,\ \beta_3 \geq 0\}$.

Now, since β_i can take any arbitrary nonnegative value and $\bar{\beta} \neq (0,0,0)$, (3) is true if and only if the coefficients of β_1, β_2 and β_3 are all positive, namely if and only if (1) holds. □

The financial interpretation of the inequalities in (1) is as follows. The first states that the call price with the highest strike is positive. The second that the call spread on the last two strikes is also positive. The third that the butterfly built on the given strikes has itself a positive price. In fact, each call option and each call spread have positive prices, as we show in the following.

Corollary 1.2. *The three option prices $C(K_1)$, $C(K_2)$ and $C(K_3)$ are positive and (strictly) decreasing in the strike:*

$$C(K_1) > C(K_2) > C(K_3) \tag{4}$$

Proof. Assume that $C(K_1) \leq C(K_2)$. Then by the second inequality in (1), we would have

$$\begin{aligned}
&(K_3 - K_2)C(K_1) - (K_3 - K_1)C(K_2) + (K_2 - K_1)C(K_3) \\
&\leq (K_3 - K_2)C(K_2) - (K_3 - K_1)C(K_2) + (K_2 - K_1)C(K_3) \\
&= (K_2 - K_1)[C(K_3) - C(K_2)] < 0
\end{aligned}$$

which contradicts the third inequality in (1). Therefore, $C(K_1) > C(K_2)$, which also implies the positivity of all prices, since $C(K_3) > 0$. □

1.2 Trading Calls, Forward Contracts and the Underlying Asset

We now assume we can also trade in forward contracts and in the underlying asset. By the put-call parity, this is equivalent to the possibility of building portfolios based both on European calls and European puts.

Proposition 1.3. *If we allow for trading also in forward contracts and in the underlying asset, the three option prices $C(K_1)$, $C(K_2)$ and $C(K_3)$ form an arbitrage-free system if and only if*

$$\begin{cases}
C(K_3) > 0 \\
C(K_2) > C(K_3) \\
(K_3 - K_2)C(K_1) - (K_3 - K_1)C(K_2) + (K_2 - K_1)C(K_3) > 0 \\
K_1[C(K_2) - S_0\, e^{-qT}] - K_2[C(K_1) - S_0\, e^{-qT}] > 0 \\
C(K_1) - S_0\, e^{-qT} + K_1\, e^{-rT} > 0
\end{cases} \tag{5}$$

Proof. The option prices $C(K_1)$, $C(K_2)$ and $C(K_3)$ form an arbitrage-free system if and only if every linear combination of the related calls with forward contracts and the underlying asset, yielding a nonnegative payoff that is positive with positive probability, has a positive price.

Since, for any (real) combinators α_1, α_2, α_3, α_4 and α_5,

$$\alpha_1 + \alpha_2 x + \sum_{i=1}^{3} \alpha_{i+2}(x - K_i)^+ = \begin{cases}
\alpha_1 + \alpha_2 x & x \leq K_1 \\
\alpha_1 + \alpha_2 x + \alpha_3(x - K_1) & K_1 < x \leq K_2 \\
\alpha_1 + \alpha_2 x + \alpha_3(x - K_1) + \alpha_4(x - K_2) & K_2 < x \leq K_3 \\
\alpha_1 - \alpha_3 K_1 - \alpha_4 K_2 - \alpha_5 K_3 + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)x & x > K_3
\end{cases}$$

we then have

$$\alpha_1 + \alpha_2 S_T + \sum_{i=1}^{3} \alpha_{i+2}(S_T - K_i)^+ \geq 0 \Leftrightarrow \begin{cases} \alpha_1 \geq 0 \\ \alpha_1 + \alpha_2 K_1 \geq 0 \\ \alpha_1 + \alpha_2 K_2 + \alpha_3 (K_2 - K_1) \geq 0 \\ \alpha_1 + \alpha_2 K_3 + \alpha_3 (K_3 - K_1) + \alpha_4 (K_3 - K_2) \geq 0 \\ \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \geq 0 \end{cases}$$

Setting $\bar{\alpha} := (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ and

$$\mathcal{A} := \left\{ \bar{\alpha} \neq (0,0,0,0,0) : \begin{cases} \alpha_1 \geq 0 \\ \alpha_1 + \alpha_2 K_1 \geq 0 \\ \alpha_1 + \alpha_2 K_2 + \alpha_3 (K_2 - K_1) \geq 0 \\ \alpha_1 + \alpha_2 K_3 + \alpha_3 (K_3 - K_1) + \alpha_4 (K_3 - K_2) \geq 0 \\ \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \geq 0 \end{cases} \right\}$$

we have therefore an arbitrage-free system if and only if

$$\alpha_1 \, e^{-rT} + \alpha_2 S_0 \, e^{-qT} + \sum_{i=1}^{3} \alpha_{i+2} C(K_i) > 0 \text{ for each } \bar{\alpha} \in \mathcal{A} \tag{6}$$

Defining the new variables vector $\bar{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$ as

$$\begin{cases} \beta_1 = \alpha_1 \\ \beta_2 = \alpha_1 + \alpha_2 K_1 \\ \beta_3 = \alpha_1 + \alpha_2 K_2 + \alpha_3 (K_2 - K_1) \\ \beta_4 = \alpha_1 + \alpha_2 K_3 + \alpha_3 (K_3 - K_1) + \alpha_4 (K_3 - K_2) \\ \beta_5 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \end{cases}$$

condition (6) is equivalent to

$$\begin{aligned} &\beta_1 \frac{C(K_1) - S_0 \, e^{-qT} + K_1 \, e^{-rT}}{K_1} + \beta_2 \frac{K_1[C(K_2) - S_0 \, e^{-qT}] - K_2[C(K_1) - S_0 \, e^{-qT}]}{K_1(K_2 - K_1)} \\ &\quad + \beta_3 \frac{(K_3 - K_2)C(K_1) - (K_3 - K_1)C(K_2) + (K_2 - K_1)C(K_3)}{(K_3 - K_2)(K_2 - K_1)} \\ &\quad + \beta_4 \frac{C(K_2) - C(K_3)}{K_3 - K_2} + \beta_5 C(K_3) > 0 \end{aligned} \tag{7}$$

for each $\bar{\beta} \in \mathcal{B} := \{\bar{\beta} \neq (0,0,0,0,0) : \beta_i \geq 0, \, i = 1, \ldots, 5\}$

Now, since β_i can take any arbitrary nonnegative value and $\bar{\beta} \neq (0,0,0,0,0)$, (7) is true if and only if the coefficients of β_i, $i = 1, \ldots, 5$, are all positive, namely if and only if (5) holds. □

Remark 1.4. *By the put-call parity, conditions (5) can be equivalently expressed in terms of put prices as follows:*

$$\begin{cases} P(K_3) + S_0\, e^{-qT} - K_3\, e^{-rT} > 0 \\ P(K_2) - K_2\, e^{-rT} > P(K_3) - K_3\, e^{-rT} \\ (K_3 - K_2)P(K_1) - (K_3 - K_1)P(K_2) + (K_2 - K_1)P(K_3) > 0 \\ K_1 P(K_2) - K_2 P(K_1) > 0 \\ P(K_1) > 0 \end{cases} \tag{8}$$

Remark 1.5. *From the proof of Proposition 1.3 and the put-call parity, we can infer that any portfolio of calls and puts (equivalently, any portfolio of calls, forward contracts and underlying asset) with nonnegative payoff can be expressed as a nonnegative linear combination of five basic derivatives with nonnegative payoff:*

1. *A call with strike K_3.*
   Payoff: $(S_T - K_3)^+$

2. *A call spread with strikes K_2 and K_3.*
   Payoff: $(S_T - K_2)^+ - (S_T - K_3)^+$

3. *A butterfly with strikes K_1, K_2 and K_3.*
   Payoff: $(K_3 - K_2)(S_T - K_1)^+ - (K_3 - K_1)(S_T - K_2)^+ + (K_2 - K_1)(S_T - K_3)^+$

4. *A butterfly put spread with strikes K_1 and K_2.*
   Payoff: $K_1(K_2 - S_T)^+ - K_2(K_1 - S_T)^+$

5. *A put with strike K_1.*
   Payoff: $(K_1 - S_T)^+$

We can now better understand why the positivity of the price of any nonnegative payoff (not identically equal to zero) is granted by the price positivity of these five basic derivatives.

Corollary 1.6. *The three call prices $C(K_1)$, $C(K_2)$ and $C(K_3)$ are positive, (strictly) decreasing in the strike and satisfy the classical no-arbitrage relation*

$$\left(S_0\, e^{-qT} - K_i\, e^{-rT}\right)^+ < C(K_i) < S_0\, e^{-qT} \quad i = 1, 2, 3 \tag{9}$$

Equivalently, the three put prices $P(K_1)$, $P(K_2)$ and $P(K_3)$ are positive, (strictly) increasing in the strike, and satisfy

$$\left(K_i\, e^{-rT} - S_0\, e^{-qT}\right)^+ < P(K_i) < K_i\, e^{-rT} \quad i = 1, 2, 3$$

Proof. As to the left inequality in (9), by Corollary 1.2 and the last two inequalities in (5), we just have to show that $C(K_3) - S_0\, e^{-qT} + K_3\, e^{-rT} > 0$. To this end we prove, by contradiction, that

$$C(K_3) - S_0\, e^{-qT} + K_3\, e^{-rT} > C(K_2) - S_0\, e^{-qT} + K_2\, e^{-rT} > 0$$

Assume $C(K_3) + K_3\, e^{-rT} \le C(K_2) + K_2\, e^{-rT}$. Then by the forth inequality in (5), we would have

$$\begin{aligned}
&(K_3 - K_2)C(K_1) - (K_3 - K_1)C(K_2) + (K_2 - K_1)C(K_3)\\
&\le (K_3 - K_2)C(K_1) - (K_3 - K_1)C(K_2) + (K_2 - K_1)[C(K_2) - (K_3 - K_2)\, e^{-rT}]\\
&= (K_3 - K_2)[C(K_1) - C(K_2) - (K_2 - K_1)\, e^{-rT}] < 0
\end{aligned}$$

which contradicts the third inequality in (5).

The right inequality in (9) is proved by considering the portfolio with weights $\bar{\alpha} = (0, 1, 0, 0, -1)$. The corresponding $\bar{\beta}$ is $\bar{\beta} = (0, K_1, K_2, K_3, 0)$ so that we can write

$$\begin{aligned}
S_0\, e^{-qT} - C(K_3) =& K_1 \frac{K_1[C(K_2) - S_0\, e^{-qT}] - K_2[C(K_1) - S_0\, e^{-qT}]}{K_1(K_2 - K_1)}\\
&+ K_2 \frac{(K_3 - K_2)C(K_1) - (K_3 - K_1)C(K_2) + (K_2 - K_1)C(K_3)}{(K_3 - K_2)(K_2 - K_1)}\\
&+ K_3 \frac{C(K_2) - C(K_3)}{K_3 - K_2} > 0
\end{aligned}$$

The statements on the put prices can be proved in a similar fashion to those of the corresponding calls. □

1.3 A general finite system

We finally hint at the general case of an options market where an arbitrary, but finite, number N of calls is given. For $N = 3$, we obviously recover the previously considered case.

The quoted strikes are denoted by K_i, $i = 1, \ldots, N$ and the associated call prices by $C(K_i)$, $i = 1, \ldots, N$.

Applying the same reasoning of the previous section, we have that the option prices $C(K_i)$ form an arbitrage-free system if and only if every linear combination of the related calls with forward contracts and the underlying asset yielding a nonnegative payoff that is positive with positive probability, has a positive price.

Since, for any (real) combinators α_i, $i = 1, \ldots, N$, and $j = 1, \ldots, N-1$,

$$\alpha_1 + \alpha_2 x + \sum_{i=1}^{N} \alpha_{i+2}(x - K_i)^+ = \begin{cases} \alpha_1 + \alpha_2 x & x \le K_1 \\ \alpha_1 + \alpha_2 x + \sum_{i=1}^{j} \alpha_{i+2}(x - K_i) & K_j < x \le K_{j+1} \\ \alpha_1 + \alpha_2 x + \sum_{i=1}^{N} \alpha_{i+2}(x - K_i) & x > K_N \end{cases}$$

we then have

$$\alpha_1 + \alpha_2 S_T + \sum_{i=1}^{N} \alpha_{i+2}(S_T - K_i)^+ \ge 0 \Leftrightarrow \begin{cases} \alpha_1 \ge 0 \\ \alpha_1 + \alpha_2 K_1 \ge 0 \\ \alpha_1 + \alpha_2 K_{j+1} + \sum_{i=1}^{j} \alpha_{i+2}(K_{j+1} - K_i) \ge 0 \\ \sum_{i=0}^{N} \alpha_{i+2} \ge 0 \end{cases}$$

Setting $\bar{\alpha} := (\alpha_1, \dots, \alpha_N)$ and

$$\mathcal{A} := \left\{ \bar{\alpha} \neq (0, \dots, 0) : \begin{cases} \alpha_1 \geq 0 \\ \alpha_1 + \alpha_2 K_1 \geq 0 \\ \alpha_1 + \alpha_2 K_{j+1} + \sum_{i=1}^{j} \alpha_{i+2}(K_{j+1} - K_i) \geq 0, \ j = 1, \dots, N-1 \\ \sum_{i=0}^{N} \alpha_{i+2} \geq 0 \end{cases} \right\}$$

we have an arbitrage-free system if and only if

$$\alpha_1 e^{-rT} + \alpha_2 S_0 e^{-qT} + \sum_{i=1}^{N} \alpha_{i+2} C(K_i) > 0 \text{ for each } \bar{\alpha} \in \mathcal{A} \tag{10}$$

We now define

$$A = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & K_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & K_2 & K_2 - K_1 & 0 & \cdots & \cdots & 0 \\ 1 & K_3 & K_3 - K_1 & K_3 - K_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 \end{pmatrix}$$

and the new variables vector $\bar{\beta} = (\beta_1, \dots, \beta_N)$ as $\bar{\beta}' = A\bar{\alpha}'$ so that

$$\begin{cases} \beta_1 = \alpha_1 \\ \beta_2 = \alpha_1 + \alpha_2 K_1 \\ \vdots \\ \beta_{j+2} = \alpha_1 + \alpha_2 K_{j+1} + \sum_{i=1}^{j} \alpha_{i+2}(K_{j+1} - K_i) \\ \vdots \\ \beta_{N+2} = \sum_{i=0}^{N} \alpha_{i+2} \end{cases}$$

Noting that matrix A is invertible since

$$\det(A) = K_1 \prod_{j=1}^{N-1} (K_{j+1} - K_j) > 0$$

we then have the following.

Proposition 1.7. *The option prices $C(K_1), C(K_2), \dots, C(K_N)$ form an arbitrage-free system if and only if*

$$\bar{v} A^{-1} > 0 \tag{11}$$

where

$$\bar{v} := \left(e^{-rT}, S_0 e^{-qT}, C(K_1), \dots, C(K_N) \right)$$

Proof. Condition (10) can be written as

$$\bar{v}\bar{\alpha}' > 0 \text{ for each } \bar{\alpha} \in \mathcal{A}$$

By definition of $\bar{\beta}$ this is equivalent to

$$\bar{v}A^{-1}\bar{\beta}' > 0 \text{ for each } \bar{\beta} \in \mathcal{B} := \{\bar{\beta} \neq (0, \ldots, 0) : \beta_i \geq 0,\, i = 1, \ldots, N\}$$

Since β_i can take any arbitrary nonnegative value and $\bar{\beta}$ can not be the null vector, this is true if and only if the coefficients of β_i, $i = 1, \ldots, N$, are all positive, namely if and only if (11) holds. □

References

[1] Black, F. and Scholes, M. (1973) The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* 81, 637-659.

[2] Cox, J. and Rubinstein, M. (1985) Options Markets, Prentice-Hall.