

Pricing Inflation-Indexed Derivatives

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Abstract

In this article, we start by briefly reviewing the approach proposed by Jarrow and Yildirim (2003) for modelling inflation and nominal rates in a consistent way. Their methodology is applied to the pricing of general inflation-indexed swaps and options. We then introduce two different market model approaches to price inflation swaps, caps and floors. Analytical formulas are explicitly derived. Finally, an example of calibration to swap market data is considered.

1 Introduction

European governments have been issuing inflation-indexed bonds since the beginning of the 80's, but it is only in the very last years that these bonds, and inflation-indexed derivatives in general, have become more and more popular.

Inflation is defined in terms of the percentage increments of a reference index, the Consumer Price Index (CPI), which is a representative basket of goods and services. The evolution of two major European and American inflation indices from September 2001 to July 2004 is shown in Figure 1.

In theory, but also in practice, inflation can become negative, so that, to preserve positivity of coupons, the inflation rate is typically floored at zero, thus implicitly offering a zero-strike floor in conjunction with the “pure” inflation-linked bond.¹

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¹A comprehensive guide to inflation-indexed derivatives is that of The Royal Bank of Scotland (2003).

Floors with low strikes are the most actively traded options on inflation rates. Other extremely popular derivatives are inflation-indexed swaps, where the inflation rate is either paid on an annual basis or with a single amount at the swap maturity.

All these inflation-indexed derivatives require a specific model to be valued. Their pricing has been tackled, among others, by Barone and Castagna (1997) and Jarrow and Yildirim (2003) (JY), who proposed similar frameworks based on a foreign-currency analogy.² In both articles, what is modelled is the evolution of the instantaneous nominal and real rates and of the CPI, which is interpreted as the “exchange rate” between the nominal and real economies. In this setting, the valuation of an inflation-indexed payoff becomes equivalent to that of a cross-currency interest rate derivative.³

The real rate one models must be intended as the “expected real rate” for the related future interval, or better as the real rate we can lock in by suitably trading in inflation swaps. The true real rate will be only known at the end of the corresponding period, as soon as the value of the CPI at that time is known. This is why there is no redundancy in modelling both this real rate and the inflation index together with the nominal rate.

The purpose of this article is to price analytically, and consistently with no arbitrage, inflation-indexed swaps and options. We first apply the JY model in its equivalent short-rate formulation and derive formulas for the new type of products we consider. We then introduce two different market models for pricing swaps and, for simplicity, stick only to the second one when pricing options.⁴ The advantage of our market-model approaches is in terms of a better understanding of the model parameters and of a more accurate calibration to market data.

The article is organized as follows. In Section 2 the JY model is briefly reviewed and reformulated in terms of instantaneous short rates. Section 3 introduces inflation-indexed swaps and values them under the JY model and two different market models. Sections 4 and 5 deal with the pricing of inflation-indexed caplets (floorlets) and caps (floors). Section 6 considers an example of calibration to real market swap data. Section 7 concludes the paper.

2 The JY model

We here briefly review the approach proposed by Jarrow and Yildirim (2003) for modelling inflation and nominal rates. Their methodology is based on a foreign-currency analogy, according to which real rates are viewed as interest rates in the real (i.e. foreign) economy and the CPI is nothing but the exchange rate between the nominal (i.e. domestic) and real “currencies”.

²Other references for the pricing of inflation-indexed derivatives are van Bezooyen et al. (1997), Hughston (1998) and Cairns (2000).

³The modelling of stochastic interest rates in a multi-currency setting has been tackled, among others, by Andreasen (1995), Frey and Sommer (1996), Flesaker and Hughston (1996), Rogers (1997) and Schlögl (2002).

⁴Our second market model is equivalent to, but independently derived from, that of Belgrade and Benhamou (2004) and Belgrade et al. (2004).

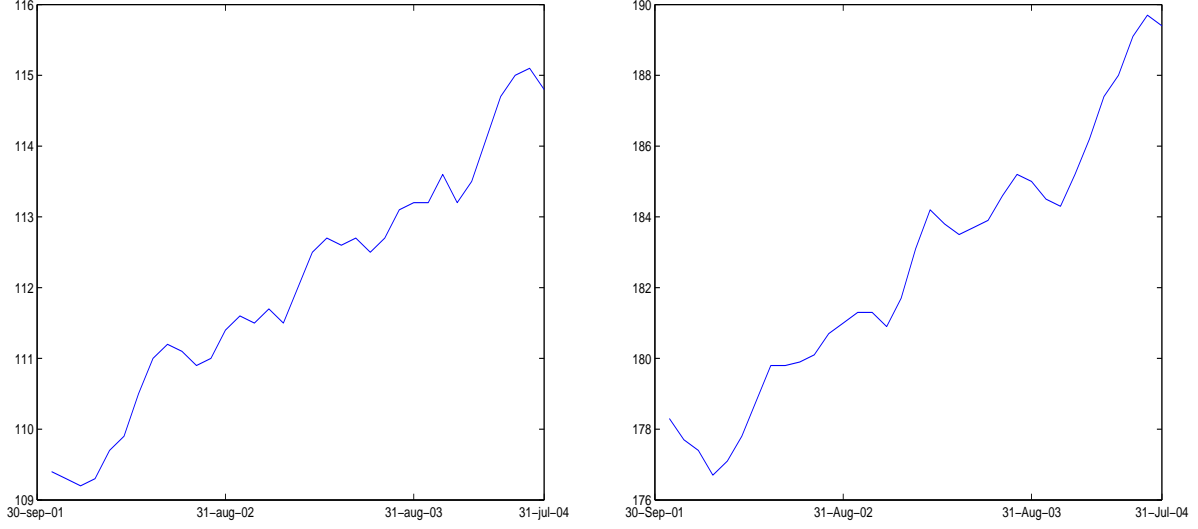


Figure 1: Left: EUR CPI Unrevised Ex-Tobacco. Right: USD CPI Urban Consumers NSA. Monthly closing values from 30-Sep-01 to 21-Jul-04.

Under the real-world probability space (Ω, \mathcal{F}, P) , with associated filtration \mathcal{F}_t , Jarrow and Yildirim considered the following evolution for the nominal and real instantaneous forward rates and for the CPI

$$\begin{aligned} df_n(t, T) &= \alpha_n(t, T) dt + \varsigma_n(t, T) dW_n^P(t) \\ df_r(t, T) &= \alpha_r(t, T) dt + \varsigma_r(t, T) dW_r^P(t) \\ dI(t) &= I(t)\mu(t) dt + \sigma_I I(t) dW_I^P(t) \end{aligned}$$

with $I(0) = I_0 > 0$, and

$$f_x(0, T) = f_x^M(0, T), \quad x \in \{n, r\},$$

where

- (W_n^P, W_r^P, W_I^P) is a Brownian motion with correlations $\rho_{n,r}$, $\rho_{n,I}$ and $\rho_{r,I}$;
- α_n , α_r and μ are adapted processes;
- ς_n and ς_r are deterministic functions;
- σ_I is a positive constant;
- $f_n^M(0, T)$ and $f_r^M(0, T)$ are, respectively, the nominal and real instantaneous forward rates observed in the market at time 0 for maturity T .

The term structures of discount factors, at time t , for the nominal and real economies are respectively given by $T \mapsto P_n(t, T)$ and $T \mapsto P_r(t, T)$ for $T \geq t$. Given the future times

T_{i-1} and T_i , the related forward rates, at time t , are defined by

$$F_x(t; T_{i-1}, T_i) = \frac{P_x(t, T_{i-1}) - P_x(t, T_i)}{\tau_i P_x(t, T_i)}, \quad x \in \{n, r\},$$

where τ_i is the year fraction for the interval $[T_{i-1}, T_i]$, which is assumed to be the same for both nominal and real rates.

We denote by Q_n and Q_r the nominal and real risk-neutral measures, respectively, and by E_x the expectation associated with Q_x , $x \in \{n, r\}$.

We denote the nominal and real instantaneous short rates, respectively, by

$$\begin{aligned} n(t) &= f_n(t, t) \\ r(t) &= f_r(t, t). \end{aligned}$$

Choosing the forward rate volatilities as

$$\begin{aligned} \varsigma_n(t, T) &= \sigma_n e^{-a_n(T-t)}, \\ \varsigma_r(t, T) &= \sigma_r e^{-a_r(T-t)}, \end{aligned}$$

where σ_n , σ_r , a_n and a_r are positive constants, and using the equivalent formulation in terms of instantaneous short rates, we have the following (see Jarrow and Yildirim, 2003).

Proposition 2.1. *The Q_n -dynamics of the instantaneous nominal rate, the instantaneous real rate and the CPI are*

$$\begin{aligned} dn(t) &= [\vartheta_n(t) - a_n n(t)] dt + \sigma_n dW_n(t) \\ dr(t) &= [\vartheta_r(t) - \rho_{r,I} \sigma_I \sigma_r - a_r r(t)] dt + \sigma_r dW_r(t) \\ dI(t) &= I(t)[n(t) - r(t)] dt + \sigma_I I(t) dW_I(t), \end{aligned} \tag{1}$$

where (W_n, W_r, W_I) is a Brownian motion with correlations $\rho_{n,r}$, $\rho_{n,I}$ and $\rho_{r,I}$, and $\vartheta_n(t)$ and $\vartheta_r(t)$ are deterministic functions to be used to exactly fit the term structures of nominal and real rates, respectively, i.e.:

$$\vartheta_x(t) = \frac{\partial f_x(0, t)}{\partial T} + a_x f_x(0, t) + \frac{\sigma_x^2}{2a_x} (1 - e^{-2a_x t}), \quad x \in \{n, r\},$$

where $\frac{\partial f_x}{\partial T}$ denotes partial derivative of f_x with respect to its second argument.

Jarrow and Yildirim thus assumed that both nominal and real (instantaneous) rates are normally distributed under their respective risk-neutral measures. They then proved that the real rate r is still an Ornstein-Uhlenbeck process under the nominal measure Q_n , and that the inflation index $I(t)$, at each time t , is lognormally distributed under Q_n , since we can write, for each $t < T$,

$$I(T) = I(t) e^{\int_t^T [n(u) - r(u)] du - \frac{1}{2} \sigma_I^2 (T-t) + \sigma_I (W_I(T) - W_I(t))}. \tag{2}$$

In the following sections, we will apply the JY model to the valuation of inflation-indexed swaps and caps.

3 Inflation-Indexed Swaps

Given a set of dates T_1, \dots, T_M , an Inflation-Indexed Swap (IIS) is a swap where, on each payment date, Party A pays Party B the inflation rate over a predefined period, while Party B pays Party A a fixed rate. The inflation rate is calculated as the percentage return of the CPI index over the time interval it applies to. Two are the main IIS traded in the market: the zero coupon (ZC) swap and the year-on-year (YY) swap.

In the ZCIIS, at the final time T_M , assuming $T_M = M$ years, Party B pays Party A the fixed amount

$$N[(1 + K)^M - 1], \quad (3)$$

where K and N are, respectively, the contract fixed rate and nominal value. In exchange for this fixed payment, Party A pays Party B, at the final time T_M , the floating amount

$$N \left[\frac{I(T_M)}{I_0} - 1 \right]. \quad (4)$$

In the YYIIS, at each time T_i , Party B pays Party A the fixed amount

$$N\varphi_i K,$$

where φ_i is the contract fixed-leg year fraction for the interval $[T_{i-1}, T_i]$, while Party A pays Party B the (floating) amount

$$N\psi_i \left[\frac{I(T_i)}{I(T_{i-1})} - 1 \right], \quad (5)$$

where ψ_i is the floating-leg year fraction for the interval $[T_{i-1}, T_i]$, $T_0 := 0$ and N is again the contract nominal value.

Both ZC and YY swaps are quoted, in the market, in terms of the corresponding fixed rate K . The ZCIIS and YYIIS (mid) fixed-rate quotes in the Euro market on October 7th 2004 are shown in Figure 2, for maturities up to twenty years. The reference CPI is the Euro-zone ex-tobacco index.

Standard no-arbitrage pricing theory implies that the value at time t , $0 \leq t < T_M$, of the inflation-indexed leg of the ZCIIS is

$$\mathbf{ZCIIS}(t, T_M, I_0, N) = NE_n \left\{ e^{-\int_t^{T_M} n(u) du} \left[\frac{I(T_M)}{I_0} - 1 \right] \middle| \mathcal{F}_t \right\}, \quad (6)$$

where \mathcal{F}_t denotes the σ -algebra generated by the relevant underlying processes up to time t .

By the foreign-currency analogy, the nominal price of a real zero-coupon bond equals the nominal price of the contract paying off one unit of the CPI index at the bond maturity. In formulas, for each $t < T$:

$$I(t)P_r(t, T) = I(t)E_r \left\{ e^{-\int_t^T r(u) du} \middle| \mathcal{F}_t \right\} = E_n \left\{ e^{-\int_t^T n(u) du} I(T) \middle| \mathcal{F}_t \right\}. \quad (7)$$

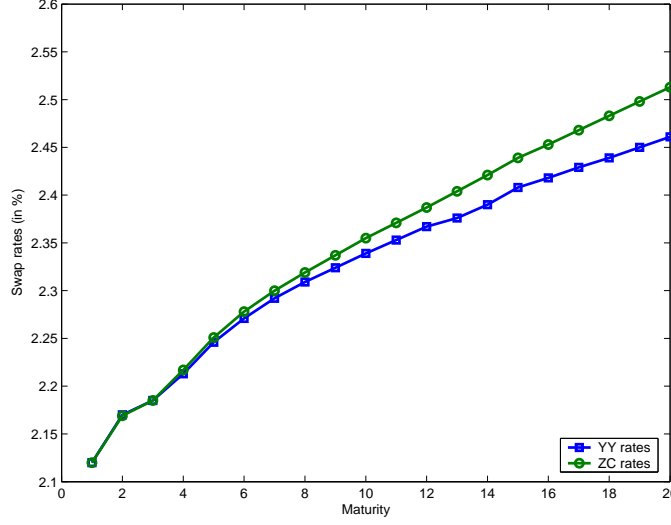


Figure 2: Euro inflation swap rates as of October 7, 2004.

Therefore, (6) becomes

$$\mathbf{ZCIIS}(t, T_M, I_0, N) = N \left[\frac{I(t)}{I_0} P_r(t, T_M) - P_n(t, T_M) \right], \quad (8)$$

which at time $t = 0$ simplifies to

$$\mathbf{ZCIIS}(0, T_M, N) = N [P_r(0, T_M) - P_n(0, T_M)]. \quad (9)$$

Formulas (8) and (9) yield model-independent prices, which are not based on specific assumptions on the evolution of the interest rate market, but simply follow from the absence of arbitrage. This result is extremely important since it enables us to strip, with no ambiguity, real zero-coupon bond prices from the quoted prices of zero-coupon inflation-indexed swaps.

In fact, the market quotes values of $K = K(T_M)$ for some given maturities T_M , so that equating (9) with the (nominal) present value of (3), and getting the discount factor $P_n(0, T_M)$ from the current (nominal) zero-coupon curve, we can solve for the unknown $P_r(0, T_M)$. We thus obtain the discount factor for maturity T_M in the real economy:⁵

$$P_r(0, T_M) = P_n(0, T_M)(1 + K(T_M))^M.$$

A different story applies to the valuation of a YYIIS. Notice, in fact, that the value at time $t < T_i$ of the payoff (5) at time T_i is

$$\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N \psi_i E_n \left\{ e^{-\int_t^{T_i} n(u) du} \left[\frac{I(T_i)}{I(T_{i-1})} - 1 \right] \middle| \mathcal{F}_t \right\}, \quad (10)$$

⁵The real discount factors for intermediate maturities can be inferred by taking into account the typical seasonality effects in inflation.

which, assuming $t < T_{i-1}$ (otherwise we reduce the calculation to the previous case), can be calculated as

$$N\psi_i E_n \left\{ e^{-\int_t^{T_{i-1}} n(u) du} E_n \left[e^{-\int_{T_{i-1}}^{T_i} n(u) du} \left(\frac{I(T_i)}{I(T_{i-1})} - 1 \right) \middle| \mathcal{F}_{T_{i-1}} \right] \middle| \mathcal{F}_t \right\}. \quad (11)$$

The inner expectation is nothing but $\mathbf{ZCIIS}(T_{i-1}, T_i, I(T_{i-1}), 1)$, so that we obtain

$$\begin{aligned} & N\psi_i E_n \left\{ e^{-\int_t^{T_{i-1}} n(u) du} [P_r(T_{i-1}, T_i) - P_n(T_{i-1}, T_i)] \middle| \mathcal{F}_t \right\} \\ &= N\psi_i E_n \left\{ e^{-\int_t^{T_{i-1}} n(u) du} P_r(T_{i-1}, T_i) \middle| \mathcal{F}_t \right\} - N\psi_i P_n(t, T_i). \end{aligned} \quad (12)$$

The last expectation can be viewed as the nominal price of a derivative paying off, in nominal units, the real zero-coupon bond price $P_r(T_{i-1}, T_i)$ at time T_{i-1} . If real rates were deterministic, then this price would simply be the present value, in nominal terms, of the forward price of the real bond. In this case, in fact, we would have:

$$E_n \left\{ e^{-\int_t^{T_{i-1}} n(u) du} P_r(T_{i-1}, T_i) \middle| \mathcal{F}_t \right\} = P_r(T_{i-1}, T_i) P_n(t, T_{i-1}) = \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} P_n(t, T_{i-1}).$$

In practice, however, real rates are stochastic and the expected value in (12) is model dependent. For instance, under dynamics (1), the forward price of the real bond must be corrected by a factor depending on both the nominal and real interest rates volatilities and on the respective correlation. This is explained in the following.

3.1 Pricing with the JY model

Denoting by Q_n^T the nominal T -forward measure for a general maturity T and by E_n^T the associated expectation, we can write:

$$\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i P_n(t, T_{i-1}) E_n^{T_{i-1}} \left\{ P_r(T_{i-1}, T_i) \middle| \mathcal{F}_t \right\} - N\psi_i P_n(t, T_i). \quad (13)$$

Remembering the zero-coupon bond price formula in the Hull and White (1994) model:

$$\begin{aligned} P_r(t, T) &= A_r(t, T) e^{-B_r(t, T)r(t)}, \\ B_r(t, T) &= \frac{1}{a_r} [1 - e^{-a_r(T-t)}], \\ A_r(t, T) &= \frac{P_r^M(0, T)}{P_r^M(0, t)} \exp \left\{ B_r(t, T) f_r^M(0, t) - \frac{\sigma_r^2}{4a_r} (1 - e^{-2a_r t}) B_r(t, T)^2 \right\}, \end{aligned}$$

and noting that, by the change-of-numeraire technique developed by Geman et al. (1995),⁶ the real instantaneous rate evolves under $Q_n^{T_{i-1}}$ according to

$$dr(t) = [-\rho_{n,r} \sigma_n \sigma_r B_n(t, T_{i-1}) + \vartheta_r(t) - \rho_{r,I} \sigma_I \sigma_r - a_r r(t)] dt + \sigma_r dW_r^{T_{i-1}}(t) \quad (14)$$

⁶See also the change-of-numeraire toolkit in Brigo and Mercurio (2001).

with $W_r^{T_{i-1}}$ a $Q_n^{T_{i-1}}$ -Brownian motion, we have that the real bond price $P_r(T_{i-1}, T_i)$ is lognormally distributed under $Q_n^{T_{i-1}}$, since $r(T_{i-1})$ is still a normal random variable under this (nominal) forward measure. After some tedious, but straightforward, algebra we finally obtain

$$\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i P_n(t, T_{i-1}) \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} e^{C(t, T_{i-1}, T_i)} - N\psi_i P_n(t, T_i), \quad (15)$$

where

$$C(t, T_{i-1}, T_i) = \sigma_r B_r(T_{i-1}, T_i) \left[B_r(t, T_{i-1}) \left(\rho_{r,I} \sigma_I - \frac{1}{2} \sigma_r B_r(t, T_{i-1}) \right) + \frac{\rho_{n,r} \sigma_n}{a_n + a_r} (1 + a_r B_n(t, T_{i-1})) \right] - \frac{\rho_{n,r} \sigma_n}{a_n + a_r} B_n(t, T_{i-1}).$$

The expectation of a real zero-coupon bond price under a nominal forward measure, in the JY model, is thus equal to the current forward price of the real bond multiplied by a correction factor, which depends on the (instantaneous) volatilities of the nominal rate, the real rate and the CPI, on the (instantaneous) correlation between nominal and real rates, and on the (instantaneous) correlation between the real rate and the CPI.

The exponential of C is the correction term we mentioned above. This term accounts for the stochasticity of real rates and, indeed, vanishes for $\sigma_r = 0$.

The value at time t of the inflation-indexed leg of the swap is simply obtained by summing up the values of all floating payments. We thus get

$$\begin{aligned} \mathbf{YYIIS}(t, \mathcal{T}, \Psi, N) = & N\psi_{\iota(t)} \left[\frac{I(t)}{I(T_{\iota(t)-1})} P_r(t, T_{\iota(t)}) - P_n(t, T_{\iota(t)}) \right] \\ & + N \sum_{i=\iota(t)+1}^M \psi_i \left[P_n(t, T_{i-1}) \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} e^{C(t, T_{i-1}, T_i)} - P_n(t, T_i) \right], \end{aligned} \quad (16)$$

where we set $\mathcal{T} := \{T_1, \dots, T_M\}$, $\Psi := \{\psi_1, \dots, \psi_M\}$ and $\iota(t) = \min\{i : T_i > t\}$,⁷ and where the first payment after time t has been priced according to (8). In particular at $t = 0$,

$$\begin{aligned} \mathbf{YYIIS}(0, \mathcal{T}, \Psi, N) = & N\psi_1 [P_r(0, T_1) - P_n(0, T_1)] \\ & + N \sum_{i=2}^M \psi_i \left[P_n(0, T_{i-1}) \frac{P_r(0, T_i)}{P_r(0, T_{i-1})} e^{C(0, T_{i-1}, T_i)} - P_n(0, T_i) \right] \\ = & N \sum_{i=1}^M \psi_i P_n(0, T_i) \left[\frac{1 + \tau_i F_n(0; T_{i-1}, T_i)}{1 + \tau_i F_r(0; T_{i-1}, T_i)} e^{C(0, T_{i-1}, T_i)} - 1 \right]. \end{aligned} \quad (17)$$

The advantage of using Gaussian models for nominal and real rates is clear as far as analytical tractability is concerned. However, the possibility of negative rates and the difficulty

⁷By definition, $T_{\iota(t)-1} \leq t < T_{\iota(t)}$.

in estimating historically the real rate parameters push us to investigate alternative approaches. To this end, we will propose, in the following, two different market models for the valuation of a YYIS and inflation-indexed options.

3.2 Pricing with a first market model

For an alternative pricing of the above YYIS, we notice that we can change measure and re-write the expectation in (13) as

$$\begin{aligned} P_n(t, T_{i-1}) E_n^{T_{i-1}} \{ P_r(T_{i-1}, T_i) | \mathcal{F}_t \} &= P_n(t, T_i) E_n^{T_i} \left\{ \frac{P_r(T_{i-1}, T_i)}{P_n(T_{i-1}, T_i)} | \mathcal{F}_t \right\} \\ &= P_n(t, T_i) E_n^{T_i} \left\{ \frac{1 + \tau_i F_n(T_{i-1}; T_{i-1}, T_i)}{1 + \tau_i F_r(T_{i-1}; T_{i-1}, T_i)} | \mathcal{F}_t \right\}, \end{aligned} \quad (18)$$

which can be calculated as soon as we specify the distribution of both forward rates under the nominal T_i -forward measure.

It seems natural, therefore, to resort to a (lognormal) LIBOR model,⁸ which postulates the evolution of simply-compounded forward rates, namely the variables that explicitly enter the last expectation.

Since $I(t)P_r(t, T_i)$ is the price of an asset in the nominal economy, we have that the forward CPI

$$\mathcal{I}_i(t) := I(t) \frac{P_r(t, T_i)}{P_n(t, T_i)}$$

is a martingale under $Q_n^{T_i}$ by the definition itself of $Q_n^{T_i}$. Assuming lognormal dynamics for \mathcal{I}_i ,

$$d\mathcal{I}_i(t) = \sigma_{I,i} \mathcal{I}_i(t) dW_i^I(t), \quad (19)$$

where $\sigma_{I,i}$ is a positive constant and W_i^I is a $Q_n^{T_i}$ -Brownian motion, and assuming also that both nominal and real forward rates follow a lognormal LIBOR market model, the analogy with cross-currency derivatives pricing implies that the dynamics of $F_n(\cdot; T_{i-1}, T_i)$ and $F_r(\cdot; T_{i-1}, T_i)$ under $Q_n^{T_i}$ are given by (see e.g. Schlögl, 2002)

$$\begin{aligned} dF_n(t; T_{i-1}, T_i) &= \sigma_{n,i} F_n(t; T_{i-1}, T_i) dW_i^n(t), \\ dF_r(t; T_{i-1}, T_i) &= F_r(t; T_{i-1}, T_i) \left[-\rho_{I,r,i} \sigma_{I,i} \sigma_{r,i} dt + \sigma_{r,i} dW_i^r(t) \right], \end{aligned} \quad (20)$$

where $\sigma_{n,i}$ and $\sigma_{r,i}$ are positive constants, W_i^n and W_i^r are two Brownian motions with instantaneous correlation ρ_i , and $\rho_{I,r,i}$ is the instantaneous correlation between $\mathcal{I}_i(\cdot)$ and $F_r(\cdot; T_{i-1}, T_i)$, i.e. $dW_i^I(t) dW_i^r(t) = \rho_{I,r,i} dt$.⁹

⁸The LIBOR market model has been independently proposed by Brace et al. (1997), Miltersen et al. (1997) and Jamshidian (1997).

⁹Assuming that $\sigma_{I,i}$, $\sigma_{n,i}$ and $\sigma_{r,i}$ are deterministic functions of time, only slightly complicates the calculations below.

The expectation in (18) can then be easily calculated with a numerical integration by noting that, under $Q_n^{T_i}$ and conditional on \mathcal{F}_t , the pair¹⁰

$$(X_i, Y_i) = \left(\ln \frac{F_n(T_{i-1}; T_{i-1}, T_i)}{F_n(t; T_{i-1}, T_i)}, \ln \frac{F_r(T_{i-1}; T_{i-1}, T_i)}{F_r(t; T_{i-1}, T_i)} \right) \quad (21)$$

is distributed as a bivariate normal random variable with mean vector and variance-covariance matrix, respectively, given by

$$M_{X_i, Y_i} = \begin{bmatrix} \mu_{x,i}(t) \\ \mu_{y,i}(t) \end{bmatrix}, \quad V_{X_i, Y_i} = \begin{bmatrix} \sigma_{x,i}^2(t) & \rho_i \sigma_{x,i}(t) \sigma_{y,i}(t) \\ \rho_i \sigma_{x,i}(t) \sigma_{y,i}(t) & \sigma_{y,i}^2(t) \end{bmatrix}, \quad (22)$$

where

$$\begin{aligned} \mu_{x,i}(t) &= -\frac{1}{2} \sigma_{n,i}^2 (T_{i-1} - t), & \sigma_{x,i}(t) &= \sigma_{n,i} \sqrt{T_{i-1} - t}, \\ \mu_{y,i}(t) &= \left[-\frac{1}{2} \sigma_{r,i}^2 - \rho_{I,r,i} \sigma_{I,i} \sigma_{r,i} \right] (T_{i-1} - t), & \sigma_{y,i}(t) &= \sigma_{r,i} \sqrt{T_{i-1} - t}. \end{aligned}$$

It is well known that the density $f_{X_i, Y_i}(x, y)$ of (X_i, Y_i) can be decomposed as

$$f_{X_i, Y_i}(x, y) = f_{X_i|Y_i}(x, y) f_{Y_i}(y),$$

where

$$\begin{aligned} f_{X_i|Y_i}(x, y) &= \frac{1}{\sigma_{x,i}(t) \sqrt{2\pi} \sqrt{1 - \rho_i^2}} \exp \left[-\frac{\left(\frac{x - \mu_{x,i}(t)}{\sigma_{x,i}(t)} - \rho_i \frac{y - \mu_{y,i}(t)}{\sigma_{y,i}(t)} \right)^2}{2(1 - \rho_i^2)} \right] \\ f_{Y_i}(y) &= \frac{1}{\sigma_{y,i}(t) \sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{y - \mu_{y,i}(t)}{\sigma_{y,i}(t)} \right)^2 \right]. \end{aligned} \quad (23)$$

The last expectation in (18) can thus be calculated as

$$\begin{aligned} & \int_{-\infty}^{+\infty} \frac{1}{1 + \tau_i F_r(t; T_{i-1}, T_i) e^y} \left[\int_{-\infty}^{+\infty} (1 + \tau_i F_n(t; T_{i-1}, T_i) e^x) f_{X_i|Y_i}(x, y) dx \right] f_{Y_i}(y) dy \\ &= \int_{-\infty}^{+\infty} \frac{1 + \tau_i F_n(t; T_{i-1}, T_i) e^{\mu_{x,i}(t) + \rho_i \sigma_{x,i}(t) \frac{y - \mu_{y,i}(t)}{\sigma_{y,i}(t)} + \frac{1}{2} \sigma_{x,i}^2(t) (1 - \rho_i^2)}}{1 + \tau_i F_r(t; T_{i-1}, T_i) e^y} \frac{e^{-\frac{1}{2} \left(\frac{y - \mu_{y,i}(t)}{\sigma_{y,i}(t)} \right)^2}}{\sigma_{y,i}(t) \sqrt{2\pi}} dy \\ &= \int_{-\infty}^{+\infty} \frac{1 + \tau_i F_n(t; T_{i-1}, T_i) e^{\rho_i \sigma_{x,i}(t) z - \frac{1}{2} \sigma_{x,i}^2(t) \rho_i^2}}{1 + \tau_i F_r(t; T_{i-1}, T_i) e^{\mu_{y,i}(t) + \sigma_{y,i}(t) z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} dz, \end{aligned}$$

yielding:

$$\begin{aligned} & \mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) \\ &= N \psi_i P_n(t, T_i) \int_{-\infty}^{+\infty} \frac{1 + \tau_i F_n(t; T_{i-1}, T_i) e^{\rho_i \sigma_{x,i}(t) z - \frac{1}{2} \sigma_{x,i}^2(t) \rho_i^2}}{1 + \tau_i F_r(t; T_{i-1}, T_i) e^{\mu_{y,i}(t) + \sigma_{y,i}(t) z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} dz - N \psi_i P_n(t, T_i). \end{aligned} \quad (24)$$

¹⁰To lighten the notation, we simply write (X_i, Y_i) instead of $(X_i(t), Y_i(t))$.

To value the whole inflation-indexed leg of the swap some care is needed, since we cannot simply sum up the values (24) of the single floating payments. In fact, as noted by Schlögl (2002), we cannot assume that the volatilities $\sigma_{I,i}$, $\sigma_{n,i}$ and $\sigma_{r,i}$ are positive constants for all i , because there exists a precise relation between two consecutive forward CPIs and the corresponding nominal and real forward rates, namely:

$$\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} = \frac{1 + \tau_i F_n(t; T_{i-1}, T_i)}{1 + \tau_i F_r(t; T_{i-1}, T_i)}. \quad (25)$$

Clearly, if we assume that $\sigma_{I,i}$, $\sigma_{n,i}$ and $\sigma_{r,i}$ are positive constants, $\sigma_{I,i-1}$ cannot be constant as well, and its admissible values are obtained by equating the (instantaneous) quadratic variations on both sides of (25).

However, by freezing the forward rates at their time 0 value in the diffusion coefficients of the right-hand-side of (25), we can still get forward CPI volatilities that are approximately constant. For instance, in the one-factor model case,

$$\begin{aligned} \sigma_{I,i-1} &= \sigma_{I,i} + \sigma_{r,i} \frac{\tau_i F_r(t; T_{i-1}, T_i)}{1 + \tau_i F_r(t; T_{i-1}, T_i)} - \sigma_{n,i} \frac{\tau_i F_n(t; T_{i-1}, T_i)}{1 + \tau_i F_n(t; T_{i-1}, T_i)} \\ &\approx \sigma_{I,i} + \sigma_{r,i} \frac{\tau_i F_r(0; T_{i-1}, T_i)}{1 + \tau_i F_r(0; T_{i-1}, T_i)} - \sigma_{n,i} \frac{\tau_i F_n(0; T_{i-1}, T_i)}{1 + \tau_i F_n(0; T_{i-1}, T_i)}. \end{aligned}$$

Therefore, applying this “freezing” procedure for each $i < M$ starting from $\sigma_{I,M}$, or equivalently for each $i > 2$ starting from $\sigma_{I,1}$, we can still assume that the volatilities $\sigma_{I,i}$ are all constant and set to one of their admissible values. The value at time t of the inflation-indexed leg of the swap is thus given by

$$\begin{aligned} \mathbf{YYIIS}(t, \mathcal{T}, \Psi, N) &= N \psi_{\iota(t)} \left[\frac{I(t)}{I(T_{\iota(t)-1})} P_r(t, T_{\iota(t)}) - P_n(t, T_{\iota(t)}) \right] \\ &+ N \sum_{i=\iota(t)+1}^M \psi_i P_n(t, T_i) \left[\int_{-\infty}^{+\infty} \frac{1 + \tau_i F_n(t; T_{i-1}, T_i) e^{\rho_i \sigma_{x,i}(t)z - \frac{1}{2} \sigma_{x,i}^2(t) \rho_i^2}}{1 + \tau_i F_r(t; T_{i-1}, T_i) e^{\mu_{y,i}(t) + \sigma_{y,i}(t)z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - 1 \right]. \end{aligned} \quad (26)$$

In particular at $t = 0$,

$$\begin{aligned} \mathbf{YYIIS}(0, \mathcal{T}, \Psi, N) &= N \psi_1 [P_r(0, T_1) - P_n(0, T_1)] \\ &+ N \sum_{i=2}^M \psi_i P_n(0, T_i) \left[\int_{-\infty}^{+\infty} \frac{1 + \tau_i F_n(0; T_{i-1}, T_i) e^{\rho_i \sigma_{x,i}(0)z - \frac{1}{2} \sigma_{x,i}^2(0) \rho_i^2}}{1 + \tau_i F_r(0; T_{i-1}, T_i) e^{\mu_{y,i}(0) + \sigma_{y,i}(0)z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - 1 \right] \\ &= N \sum_{i=1}^M \psi_i P_n(0, T_i) \left[\int_{-\infty}^{+\infty} \frac{1 + \tau_i F_n(0; T_{i-1}, T_i) e^{\rho_i \sigma_{x,i}(0)z - \frac{1}{2} \sigma_{x,i}^2(0) \rho_i^2}}{1 + \tau_i F_r(0; T_{i-1}, T_i) e^{\mu_{y,i}(0) + \sigma_{y,i}(0)z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - 1 \right]. \end{aligned} \quad (27)$$

This YYIIS price depends on the following parameters: the (instantaneous) volatilities of nominal and real forward rates and their correlations, for each payment time T_i , $i =$

2, \dots, M; the (instantaneous) volatilities of forward inflation indices and their correlations with real forward rates, again for each $i = 2, \dots, M$.

Compared with expression (17), formula (27) looks more complicated both in terms of input parameters and in terms of the calculations involved. However, one-dimensional numerical integrations are not so cumbersome and time consuming. Moreover, as is typical in a market model, the input parameters can be determined more easily than those coming from the previous short-rate approach.¹¹ In this respect, formula (27) is preferable to (17).

As in the previous JY case, valuing a YIIS with a LIBOR market model has the drawback that the volatility of real rates may be hard to estimate, especially when resorting to a historical calibration. This is why we propose, in the following, a second market model approach, which enables us to overcome this estimation issue.

3.3 Pricing with a second market model

Applying the definition of forward CPI and using the fact that \mathcal{I}_i is a martingale under $Q_n^{T_i}$, we can also write, for $t < T_{i-1}$,

$$\begin{aligned} \text{YIIS}(t, T_{i-1}, T_i, \psi_i, N) &= N\psi_i P(t, T_i) E_n^{T_i} \left\{ \frac{I(T_i)}{I(T_{i-1})} - 1 \middle| \mathcal{F}_t \right\} \\ &= N\psi_i P(t, T_i) E_n^{T_i} \left\{ \frac{\mathcal{I}_i(T_i)}{\mathcal{I}_{i-1}(T_{i-1})} - 1 \middle| \mathcal{F}_t \right\} \\ &= N\psi_i P(t, T_i) E_n^{T_i} \left\{ \frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} - 1 \middle| \mathcal{F}_t \right\}. \end{aligned} \quad (28)$$

The dynamics of \mathcal{I}_i under $Q_n^{T_i}$ is given by (19) and an analogous evolution holds for \mathcal{I}_{i-1} under $Q_n^{T_{i-1}}$. The dynamics of \mathcal{I}_{i-1} under $Q_n^{T_i}$ can be derived by the classical change-of-numeraire technique.¹² We get:

$$d\mathcal{I}_{i-1}(t) = -\mathcal{I}_{i-1}(t) \sigma_{I,i-1} \frac{\tau_i \sigma_{n,i} F_n(t; T_{i-1}, T_i)}{1 + \tau_i F_n(t; T_{i-1}, T_i)} \rho_{I,n,i} dt + \sigma_{I,i-1} \mathcal{I}_{i-1}(t) dW_{i-1}^I(t), \quad (29)$$

where $\sigma_{I,i-1}$ is a positive constant, W_{i-1}^I is a $Q_n^{T_i}$ -Brownian motion with $dW_{i-1}^I(t) dW_i^I(t) = \rho_{I,i} dt$, and $\rho_{I,n,i}$ is the instantaneous correlation between $\mathcal{I}_{i-1}(\cdot)$ and $F_n(\cdot; T_{i-1}, T_i)$.

The evolution of \mathcal{I}_{i-1} , under $Q_n^{T_i}$, depends on the nominal rate $F_n(\cdot; T_{i-1}, T_i)$, so that the calculation of (28) is rather involved in general. To avoid unpleasant complications, like those induced by higher-dimensional integrations, we freeze the drift in (29) at its current time- t value, so that $\mathcal{I}_{i-1}(T_{i-1})$ conditional on \mathcal{F}_t is lognormally distributed also under $Q_n^{T_i}$. This leads to

$$E_n^{T_i} \left\{ \frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} \middle| \mathcal{F}_t \right\} = \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} e^{D_i(t)},$$

¹¹There exists a huge literature on calibration issues of a (lognormal) LIBOR market model. We quote, as examples, the works of Pelsser et al. (2001), Brigo and Mercurio (2001, 2002), Rebonato (2002), Schoenmakers and Coffey (2003) and Choy et al. (2004).

¹²Musiela and Rutkowski (1997) were the first to use it in a LIBOR market model setting.

where

$$D_i(t) = \sigma_{I,i-1} \left[\frac{\tau_i \sigma_{n,i} F_n(t; T_{i-1}, T_i)}{1 + \tau_i F_n(t; T_{i-1}, T_i)} \rho_{I,n,i} - \rho_{I,i} \sigma_{I,i} + \sigma_{I,i-1} \right] (T_{i-1} - t),$$

so that

$$\mathbf{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N \psi_i P_n(t, T_i) \left[\frac{P_n(t, T_{i-1}) P_r(t, T_i)}{P_n(t, T_i) P_r(t, T_{i-1})} e^{D_i(t)} - 1 \right]. \quad (30)$$

Finally, the value at time t of the inflation-indexed leg of the swap is

$$\begin{aligned} \mathbf{YYIIS}(t, \mathcal{T}, \Psi, N) = & N \psi_{\iota(t)} \left[\frac{I(t)}{I(T_{\iota(t)-1})} P_r(t, T_{\iota(t)}) - P_n(t, T_{\iota(t)}) \right] \\ & + N \sum_{i=\iota(t)+1}^M \psi_i \left[P_n(t, T_{i-1}) \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} e^{D_i(t)} - P_n(t, T_i) \right]. \end{aligned} \quad (31)$$

In particular at $t = 0$,

$$\begin{aligned} \mathbf{YYIIS}(0, \mathcal{T}, \Psi, N) = & N \psi_1 [P_r(0, T_1) - P_n(0, T_1)] \\ & + N \sum_{i=2}^M \psi_i \left[P_n(0, T_{i-1}) \frac{P_r(0, T_i)}{P_r(0, T_{i-1})} e^{D_i(0)} - P_n(0, T_i) \right] \\ = & N \sum_{i=1}^M \psi_i P_n(0, T_i) \left[\frac{1 + \tau_i F_n(0; T_{i-1}, T_i)}{1 + \tau_i F_r(0; T_{i-1}, T_i)} e^{D_i(0)} - 1 \right]. \end{aligned} \quad (32)$$

This YYIIS price depends on the following parameters: the (instantaneous) volatilities of forward inflation indices and their correlations; the (instantaneous) volatilities of nominal forward rates; the instantaneous correlations between forward inflation indices and nominal forward rates.

Expression (32) looks pretty similar to (17) and may be preferred to (27) since it combines the advantage of a fully-analytical formula with that of a market-model approach. Moreover, contrary to (27), the correction term D does not depend on the volatility of real rates.

A drawback of formula (32) is that the approximation it is based on may be rough for long maturities T_i . In fact, such a formula is exact when the correlations $\rho_{I,n,i}$ are set to zero and the terms D_i are simplified accordingly. In general, however, such correlations can have a non-negligible impact on the D_i , and non-zero values can be found when calibrating the model to YYIIS market data.

To visualize the magnitude of the correction terms D_i in the pricing formula (32), we plot in Figure 3 the values of $D_i(0)$ corresponding to setting $T_i = i$ years, $i = 2, 3, \dots, 20$, $\sigma_{I,i} = 0.006$, $\sigma_{n,i} = 0.22$, $\rho_{I,n,i} = 0.2$, $\rho_{I,i} = 0.6$, for each i , and where the forward rates $F_n(0; T_{i-1}, T_i)$ are stripped from the Euro nominal zero-coupon curve as of 7 October 2004.

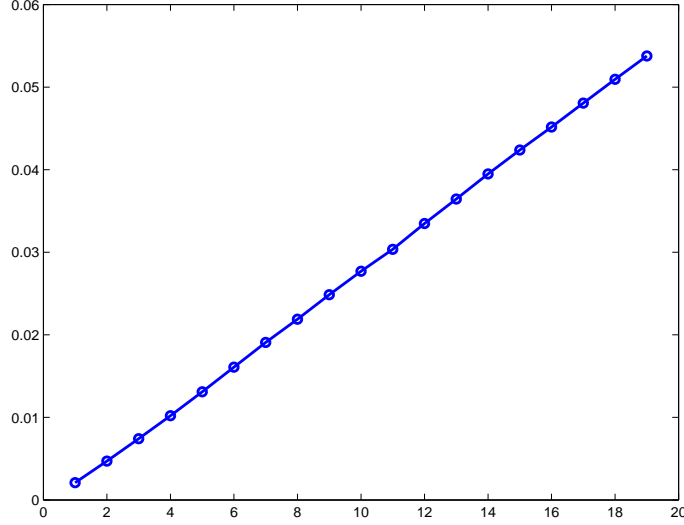


Figure 3: Plot of values $D_i(0)$, in percentage points, for $i = 2, 3, \dots, 20$.

4 Inflation-Indexed Caplets/Floorlets

An Inflation-Indexed Caplet (IIC) is a call option on the inflation rate implied by the CPI index. Analogously, an Inflation-Indexed Floorlet (IIF) is a put option on the same inflation rate. In formulas, at time T_i , the IICF payoff is

$$N\psi_i \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa \right) \right]^+, \quad (33)$$

where κ is the IICF strike, ψ_i is the contract year fraction for the interval $[T_{i-1}, T_i]$, N is the contract nominal value, and $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet.

Setting $K := 1 + \kappa$, standard no-arbitrage pricing theory implies that the value at time $t \leq T_{i-1}$ of the payoff (33) at time T_i is

$$\begin{aligned} \mathbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) &= N\psi_i E_n \left\{ e^{-\int_t^{T_i} n(u) du} \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \middle| \mathcal{F}_t \right\} \\ &= N\psi_i P_n(t, T_i) E_n^{T_i} \left\{ \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \middle| \mathcal{F}_t \right\}. \end{aligned} \quad (34)$$

The pricing of an IICF is therefore similar to that of a forward-start (cliquet) option. We now derive analytical formulas both under the JY model and under our second market model.

4.1 Pricing with the JY model

As previously mentioned, assuming Gaussian nominal and real rates leads to a CPI that is lognormally distributed under Q_n . Under this assumption, the type of distribution of

the CPI is preserved when we move to a nominal forward measure. Precisely, the ratio $I(T_i)/I(T_{i-1})$ conditional on \mathcal{F}_t is lognormally distributed also under $Q_n^{T_i}$. This implies that (34) can be calculated as soon as we know the expectation of this ratio and the variance of its logarithm. Notice, in fact, that if X is a lognormal random variable with $E(X) = m$ and $\text{Std}[\ln(X)] = v$, then

$$E\{[\omega(X - K)]^+\} = \omega m \Phi\left(\omega \frac{\ln \frac{m}{K} + \frac{1}{2}v^2}{v}\right) - \omega K \Phi\left(\omega \frac{\ln \frac{m}{K} - \frac{1}{2}v^2}{v}\right), \quad (35)$$

where Φ denotes the standard normal distribution function.

The conditional expectation of $I(T_i)/I(T_{i-1})$ is immediately obtained through the price of a YYIS, and formula (15) in particular:

$$E_n^{T_i} \left\{ \frac{I(T_i)}{I(T_{i-1})} \middle| \mathcal{F}_t \right\} = \frac{P_n(t, T_{i-1})}{P_n(t, T_i)} \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} e^{C(t, T_{i-1}, T_i)}.$$

The variance of the logarithm of the ratio can be equivalently calculated under the (nominal) risk-neutral measure.¹³ After tedious, but straightforward, calculations we get

$$\text{Var}_n^{T_i} \left\{ \ln \frac{I(T_i)}{I(T_{i-1})} \middle| \mathcal{F}_t \right\} = V^2(t, T_{i-1}, T_i),$$

where

$$\begin{aligned} & V^2(t, T_{i-1}, T_i) \\ &= \frac{\sigma_n^2}{2a_n^3} (1 - e^{-a_n(T_i - T_{i-1})})^2 [1 - e^{-2a_n(T_{i-1} - t)}] + \frac{\sigma_r^2}{2a_r^3} (1 - e^{-a_r(T_i - T_{i-1})})^2 [1 - e^{-2a_r(T_{i-1} - t)}] \\ &- 2\rho_{n,r} \frac{\sigma_n \sigma_r}{a_n a_r (a_n + a_r)} (1 - e^{-a_n(T_i - T_{i-1})}) (1 - e^{-a_r(T_i - T_{i-1})}) [1 - e^{-(a_n + a_r)(T_{i-1} - t)}] \\ &+ \sigma_I^2 (T_i - T_{i-1}) + \frac{\sigma_n^2}{a_n^2} \left[T_i - T_{i-1} + \frac{2}{a_n} e^{-a_n(T_i - T_{i-1})} - \frac{1}{2a_n} e^{-2a_n(T_i - T_{i-1})} - \frac{3}{2a_n} \right] \\ &+ \frac{\sigma_r^2}{a_r^2} \left[T_i - T_{i-1} + \frac{2}{a_r} e^{-a_r(T_i - T_{i-1})} - \frac{1}{2a_r} e^{-2a_r(T_i - T_{i-1})} - \frac{3}{2a_r} \right] \\ &- 2\rho_{n,r} \frac{\sigma_n \sigma_r}{a_n a_r} \left[T_i - T_{i-1} - \frac{1 - e^{-a_n(T_i - T_{i-1})}}{a_n} - \frac{1 - e^{-a_r(T_i - T_{i-1})}}{a_r} + \frac{1 - e^{-(a_n + a_r)(T_i - T_{i-1})}}{a_n + a_r} \right] \\ &+ 2\rho_{n,I} \frac{\sigma_n \sigma_I}{a_n} \left[T_i - T_{i-1} - \frac{1 - e^{-a_n(T_i - T_{i-1})}}{a_n} \right] - 2\rho_{r,I} \frac{\sigma_r \sigma_I}{a_r} \left[T_i - T_{i-1} - \frac{1 - e^{-a_r(T_i - T_{i-1})}}{a_r} \right]. \end{aligned}$$

¹³This is because the change of measure produces only a deterministic additive term, which has no impact in the variance calculation.

By (35), we thus have¹⁴

$$\begin{aligned}
\mathbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) &= \omega N \psi_i P_n(t, T_i) \left[\frac{P_n(t, T_{i-1})}{P_n(t, T_i)} \frac{P_r(t, T_i)}{P_r(t, T_{i-1})} e^{C(t, T_{i-1}, T_i)} \right. \\
&\cdot \Phi \left(\frac{\ln \frac{P_n(t, T_{i-1}) P_r(t, T_i)}{K P_n(t, T_i) P_r(t, T_{i-1})} + C(t, T_{i-1}, T_i) + \frac{1}{2} V^2(t, T_{i-1}, T_i)}{V(t, T_{i-1}, T_i)} \right) \\
&\left. - K \Phi \left(\frac{\ln \frac{P_n(t, T_{i-1}) P_r(t, T_i)}{K P_n(t, T_i) P_r(t, T_{i-1})} + C(t, T_{i-1}, T_i) - \frac{1}{2} V^2(t, T_{i-1}, T_i)}{V(t, T_{i-1}, T_i)} \right) \right]. \tag{36}
\end{aligned}$$

4.2 Pricing with the second market model

We now try and calculate (34) under a market model. To this end, we apply the tower property of conditional expectations to get

$$\begin{aligned}
\mathbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) \\
&= N \psi_i P_n(t, T_i) E_n^{T_i} \left\{ \frac{E_n^{T_i} \{ [\omega (I(T_i) - KI(T_{i-1}))]^+ | \mathcal{F}_{T_{i-1}} \}}{I(T_{i-1})} | \mathcal{F}_t \right\}, \tag{37}
\end{aligned}$$

where we assume that $I(T_{i-1}) > 0$.

Sticking to a market-model approach, the calculation of the outer expectation in (37) depends on whether one models forward rates or directly the forward CPI. As in Section 3.3, we here follow the latter approach, since it allows us to derive a simpler formula with less input parameters. For completeness, the former case is dealt with in the appendix.

Assuming that (19) holds and remembering that $I(T_i) = \mathcal{I}_i(T_i)$, we have

$$\begin{aligned}
E_n^{T_i} \{ [\omega (I(T_i) - KI(T_{i-1}))]^+ | \mathcal{F}_{T_{i-1}} \} &= E_n^{T_i} \{ [\omega (\mathcal{I}_i(T_i) - KI(T_{i-1}))]^+ | \mathcal{F}_{T_{i-1}} \} \\
&= \omega \mathcal{I}_i(T_{i-1}) \Phi \left(\frac{\ln \frac{\mathcal{I}_i(T_{i-1})}{KI(T_{i-1})} + \frac{1}{2} \sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i} \sqrt{T_i - T_{i-1}}} \right) \\
&\quad - \omega KI(T_{i-1}) \Phi \left(\frac{\ln \frac{\mathcal{I}_i(T_{i-1})}{KI(T_{i-1})} - \frac{1}{2} \sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i} \sqrt{T_i - T_{i-1}}} \right),
\end{aligned}$$

so that, by (37) and the definition of \mathcal{I}_{i-1} , omitting some arguments in \mathbf{IICplt} ,

$$\begin{aligned}
\mathbf{IICplt}(t) &= \omega N \psi_i P_n(t, T_i) E_n^{T_i} \left\{ \frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} \Phi \left(\frac{\ln \frac{\mathcal{I}_i(T_{i-1})}{K \mathcal{I}_{i-1}(T_{i-1})} + \frac{1}{2} \sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i} \sqrt{T_i - T_{i-1}}} \right) \right. \\
&\quad \left. - K \Phi \left(\frac{\ln \frac{\mathcal{I}_i(T_{i-1})}{K \mathcal{I}_{i-1}(T_{i-1})} - \frac{1}{2} \sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i} \sqrt{T_i - T_{i-1}}} \right) | \mathcal{F}_t \right\}. \tag{38}
\end{aligned}$$

¹⁴A similar formula is in the guide of The Royal Bank of Scotland (2003).

Remembering (29) and freezing again the drift at its time- t value, we have that under $Q_n^{T_i}$ the ratio $\mathcal{I}_i(T_{i-1})/\mathcal{I}_{i-1}(T_{i-1})$, conditional on \mathcal{F}_t , is lognormally distributed. Precisely, setting

$$V_i(t) := \sqrt{(\sigma_{I,i-1}^2 + \sigma_{I,i}^2 - 2\rho_{I,i}\sigma_{I,i-1}\sigma_{I,i})(T_{i-1} - t)}$$

we have that

$$\ln \frac{\mathcal{I}_i(T_{i-1})}{\mathcal{I}_{i-1}(T_{i-1})} | \mathcal{F}_t \sim \mathcal{N} \left(\ln \frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} + D_i(t) - \frac{1}{2} V_i^2(t), V_i^2(t) \right).$$

Straightforward algebra then leads to

$$\begin{aligned} & \mathbf{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) \\ &= \omega N \psi_i P_n(t, T_i) \left[\frac{\mathcal{I}_i(t)}{\mathcal{I}_{i-1}(t)} e^{D_i(t)} \Phi \left(\omega \frac{\ln \frac{\mathcal{I}_i(t)}{K \mathcal{I}_{i-1}(t)} + D_i(t) + \frac{1}{2} \mathcal{V}_i^2(t)}{\mathcal{V}_i(t)} \right) \right. \\ & \quad \left. - K \Phi \left(\omega \frac{\ln \frac{\mathcal{I}_i(t)}{K \mathcal{I}_{i-1}(t)} + D_i(t) - \frac{1}{2} \mathcal{V}_i^2(t)}{\mathcal{V}_i(t)} \right) \right] \\ &= \omega N \psi_i P_n(t, T_i) \left[\frac{1 + \tau_i F_n(t; T_{i-1}, T_i)}{1 + \tau_i F_r(t; T_{i-1}, T_i)} e^{D_i(t)} \Phi \left(\omega \frac{\ln \frac{1 + \tau_i F_n(t; T_{i-1}, T_i)}{K [1 + \tau_i F_r(t; T_{i-1}, T_i)]} + D_i(t) + \frac{1}{2} \mathcal{V}_i^2(t)}{\mathcal{V}_i(t)} \right) \right. \\ & \quad \left. - K \Phi \left(\omega \frac{\ln \frac{1 + \tau_i F_n(t; T_{i-1}, T_i)}{K [1 + \tau_i F_r(t; T_{i-1}, T_i)]} + D_i(t) - \frac{1}{2} \mathcal{V}_i^2(t)}{\mathcal{V}_i(t)} \right) \right], \end{aligned} \tag{39}$$

where we set

$$\mathcal{V}_i(t) = \sqrt{V_i^2(t) + \sigma_{I,i}^2 (T_i - T_{i-1})}.$$

Analogously to the YYIS prices (30) and (31), this caplet price depends on the (instantaneous) volatilities of forward inflation indices and their correlations, the (instantaneous) volatilities of nominal forward rates, and the instantaneous correlations between forward inflation indices and nominal forward rates. Therefore, formula (39) has, in terms of input parameters, all the advantages and drawbacks of the swap price (31).

The analogy with the Black and Scholes (1973) formula renders (39) quite appealing from a practical point of view, and provides a further support for the modelling of forward CPIs as geometric Brownian motions under their associated measures.

5 Inflation-Indexed Caps

An Inflation-Indexed Cap (IICap) is a stream of inflation-indexed caplets. An analogous definition holds for an Inflation-Indexed Floor (IIFloor). Given the set of dates T_0, T_1, \dots, T_M , with $T_0 = 0$, an IICapFloor pays off, at each time T_i , $1, \dots, M$,

$$N \psi_i \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa \right) \right]^+, \tag{40}$$

where κ is the IICapFloor strike.

Setting again $K := 1 + \kappa$, standard no-arbitrage pricing theory implies that the value at time 0 of the above IICapFloor is

$$\begin{aligned} \text{IICapFloor}(0, \mathcal{T}, \Psi, K, N, \omega) &= N \sum_{i=1}^M \psi_i E_n \left\{ e^{-\int_0^{T_i} n(u) du} \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \right\} \\ &= N \sum_{i=1}^M P_n(0, T_i) \psi_i E_n^{T_i} \left\{ \left[\omega \left(\frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \right\}, \end{aligned}$$

where we set again $\mathcal{T} := \{T_1, \dots, T_M\}$ and $\Psi := \{\psi_1, \dots, \psi_M\}$.

Sticking to our second market model, from (39), we immediately get

$$\begin{aligned} \text{IICapFloor}(0, \mathcal{T}, \Psi, K, N, \omega) &= \omega N \sum_{i=1}^M \psi_i P_n(0, T_i) \left[\frac{1 + \tau_i F_n(0; T_{i-1}, T_i)}{1 + \tau_i F_r(0; T_{i-1}, T_i)} e^{D_i(0)} \right. \\ &\quad \cdot \Phi \left(\omega \frac{\ln \frac{1 + \tau_i F_n(0; T_{i-1}, T_i)}{K[1 + \tau_i F_r(0; T_{i-1}, T_i)]} + D_i(0) + \frac{1}{2} \mathcal{V}_i^2(0)}{\mathcal{V}_i(0)} \right) \\ &\quad \left. - K \Phi \left(\omega \frac{\ln \frac{1 + \tau_i F_n(0; T_{i-1}, T_i)}{K[1 + \tau_i F_r(0; T_{i-1}, T_i)]} + D_i(0) - \frac{1}{2} \mathcal{V}_i^2(0)}{\mathcal{V}_i(0)} \right) \right]. \end{aligned} \quad (41)$$

6 Calibration to market data

In this section, we consider an example of calibration to Euro market data as of October 7, 2004. Precisely, we test the performance of the JY model and our two market models as far as the calibration to inflation-indexed swaps is concerned, with some model parameters being previously fitted to at-the-money (nominal) cap volatilities. The zero-coupon and year-on-year swap rates we consider are plotted in Figure 2.

As explained in Section 3, we use the zero-coupon rates to strip the current real discount factors for the relevant maturities. From these discount factors, we then derive the real forward rates that enter the pricing functions (17), (27) and (32) for the JY model, the first and the second market models, respectively.

The model parameters that best fit the given set of market data are found by minimizing the square absolute difference between model and market YYIIS fixed rates, under some constraints we introduce to avoid over-parametrization. For a given vector of model parameters \underline{p} , the model YYIIS fixed rate is defined as the rate $K = K(\underline{p})$ that renders the corresponding YYIIS a zero-value contract at the current time.

The JY formula (17) involves seven parameters: a_n , σ_n , a_r , σ_r , σ_I , $\rho_{n,r}$ and $\rho_{r,I}$. We reduce, however, this number to five, by finding a_n and σ_n through a previous calibration to the at-the-money (nominal) caps market.

The first market model formula (27) involves five parameters for each payment time from the second year onwards: $\sigma_{n,i}$, $\sigma_{r,i}$, $\sigma_{I,i}$, ρ_i , and $\rho_{I,r,i}$, for $i = 2, \dots, M = 20$. In this

case, we reduce the optimization parameters to five, c_1, c_2, \dots, c_5 , since we automatically calibrate each $\sigma_{n,i}$ to the at-the-money (nominal) cap volatilities, and set $\sigma_{r,i} = c_1/[1 + c_2(T_i - T_2)]$, $\sigma_{I,1} = c_3$, with the subsequent $\sigma_{I,i}$'s that are computed remembering (25), $\rho_i = c_4$ and $\rho_{I,r,i} = c_5$, for each $i > 1$.

Also the second market model formula (32) involves five parameters for each payment time from the second year onwards: $\sigma_{I,i-1}$, $\sigma_{n,i}$, $\rho_{I,i}$, $\rho_{I,n,i}$, and $\sigma_{I,i}$, for $i = 2, \dots, M = 20$. In this last case, we reduce the optimization parameters to four: c_1, \dots, c_4 . In fact, each $\sigma_{n,i}$ is again calibrated automatically to the at-the-money (nominal) cap volatilities. We then set $\rho_{I,i} = 1 - (1 - c_1) \exp(-c_2 T_{i-1})$, $\rho_{I,n,i} = c_3$ and $\sigma_{I,i} = c_4$, for each $i > 1$.

Our calibration results are shown in Table 1, where the three model swap rates (in percentage points) are compared with the market ones. The performance of these models

Maturity	Market	JY	MM1	MM2
1	2.120	2.120	2.120	2.120
2	2.170	2.169	2.168	2.168
3	2.185	2.186	2.186	2.184
4	2.213	2.217	2.218	2.215
5	2.246	2.250	2.250	2.247
6	2.271	2.276	2.275	2.272
7	2.292	2.296	2.295	2.293
8	2.309	2.314	2.312	2.310
9	2.324	2.324	2.322	2.320
10	2.339	2.345	2.343	2.341
11	2.353	2.358	2.356	2.355
12	2.367	2.371	2.369	2.369
13	2.383	2.385	2.383	2.383
14	2.390	2.397	2.396	2.396
15	2.408	2.410	2.410	2.410
16	2.418	2.420	2.421	2.421
17	2.429	2.430	2.431	2.432
18	2.439	2.439	2.442	2.443
19	2.450	2.448	2.453	2.454
20	2.461	2.457	2.463	2.465

Table 1: Comparison between market YYIIS fixed rates (in percentage points) and those implied by the JY model, the first market model (MM1) and the second one (MM2).

is quite satisfactory. In fact, the largest difference between a model swap rate and the corresponding market value is 0.7 basis points, which is negligible also because typical bid-ask spreads in the market are between five and ten basis points.

Even though the three models are equivalent in terms of calibration to market YYIIS rates, they can however imply quite different prices when away-from-the money derivatives

are considered. As an example, we price zero-strike floors, for maturities from four to twenty years, with the JY model and our second market model, using the parameters coming from the previous calibration. The result of this test is shown in Figure 4.

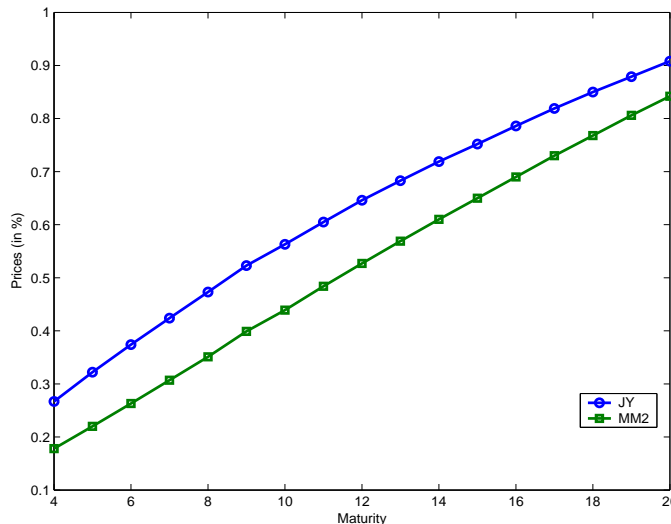


Figure 4: Comparison of zero-strike floors prices implied by the JY and second market models, for different maturities.

7 Conclusions

The classical pricing of inflation-indexed derivatives requires the modelling of both nominal and real rates and of the reference consumer price index, whose percentage returns define the inflation rate. The foreign-currency analogy allows one to view real rates as the rates in a foreign economy and to treat the CPI as an exchange rate between the domestic (nominal) and foreign (real) economies.

Assuming a Gaussian distribution for both instantaneous (nominal and real) rates, as in the Jarrow and Yildirim (2003) model, we have derived analytical formulas for inflation-indexed swaps and caps. We have also proposed two alternative market models leading to analytical formulas for these basic derivatives.

The performance of the proposed models, in terms of calibration to market data, has been tested on Euro inflation-indexed swaps. The results we obtained are quite satisfactory, given also the constraints on the model parameters we introduced to avoid over-parametrization. The comparison of the performances of different parameterizations, and the calibration to different market data are interesting subjects for future research.

Finally, it is worth mentioning that both market-model approaches allow for straightforward extensions based on forward volatility uncertainty in the spirit of Brigo, Mercurio and Rapisarda (2004) and Gatarek (2003). These extensions can be used either to incor-

porate the (nominal) caps smile/skew into the pricing of inflation-indexed derivatives or to better accommodate a possible smile in the IICapFloor market.

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8 Appendix: IICapFloor pricing with a LIBOR market model

We here show how to price a IICapFloor in the LIBOR market model of Section 3.2. To this end, we notice that, by (38) and the definition of \mathcal{I}_i

$$\begin{aligned} \text{IICplt}(t) = \omega N \psi_i P_n(t, T_i) E_n^{T_i} & \left\{ \frac{P_r(T_{i-1}, T_i)}{P_n(T_{i-1}, T_i)} \Phi \left(\omega \frac{\ln \frac{P_r(T_{i-1}, T_i)}{K P_n(T_{i-1}, T_i)} + \frac{1}{2} \sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i} \sqrt{T_i - T_{i-1}}} \right) \right. \\ & \left. - K \Phi \left(\omega \frac{\ln \frac{P_r(T_{i-1}, T_i)}{K P_n(T_{i-1}, T_i)} - \frac{1}{2} \sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i} \sqrt{T_i - T_{i-1}}} \right) \middle| \mathcal{F}_t \right\}, \end{aligned} \quad (42)$$

and that the expectation in (42) can be rewritten as

$$\begin{aligned} E_n^{T_i} & \left\{ \frac{1 + \tau_i F_n(T_{i-1}; T_{i-1}, T_i)}{1 + \tau_i F_r(T_{i-1}; T_{i-1}, T_i)} \Phi \left(\omega \frac{\ln \frac{1 + \tau_i F_n(T_{i-1}; T_{i-1}, T_i)}{K [1 + \tau_i F_r(T_{i-1}; T_{i-1}, T_i)]} + \frac{1}{2} \sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i} \sqrt{T_i - T_{i-1}}} \right) \right. \\ & \left. - K \Phi \left(\omega \frac{\ln \frac{1 + \tau_i F_n(T_{i-1}; T_{i-1}, T_i)}{K [1 + \tau_i F_r(T_{i-1}; T_{i-1}, T_i)]} - \frac{1}{2} \sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i} \sqrt{T_i - T_{i-1}}} \right) \middle| \mathcal{F}_t \right\}. \end{aligned} \quad (43)$$

Following the approach of Section 3.2, we assume that nominal and real forward rates evolve according to (20). The expectation in (43) can then be easily calculated with a numerical integration by noting again that the pair (21) is distributed as a bivariate normal random variable with mean vector and variance-covariance matrix given by (22).

The dimensionality of the problem to solve can, however, be reduced by assuming deterministic real rates, as we do in the following.¹⁵

Under deterministic real rates, the future LIBOR value $F_r(T_{i-1}; T_{i-1}, T_i)$ is simply equal

¹⁵We have previously seen that both the volatility of real rates and their correlation with nominal rates can play an important role in the pricing of inflation-linked derivatives. We here assume deterministic real rates, for simplicity, even though these parameters should be explicitly taken into account, in general.

to the current forward rate $F_r(0; T_{i-1}, T_i)$, so that we can write

$$\begin{aligned} \mathbf{ICap}(0, \mathcal{T}, \Psi, K, N, \omega) = & \omega N \sum_{i=1}^M \psi_i P_n(0, T_i) E_n^{T_i} \left\{ \frac{1 + \tau_i F_n(T_{i-1}; T_{i-1}, T_i)}{1 + \tau_i F_r(0; T_{i-1}, T_i)} \right. \\ & \cdot \Phi \left(\omega \frac{\ln \frac{1 + \tau_i F_n(T_{i-1}; T_{i-1}, T_i)}{K[1 + \tau_i F_r(0; T_{i-1}, T_i)]} + \frac{1}{2} \sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i} \sqrt{T_i - T_{i-1}}} \right) \\ & \left. - K \Phi \left(\omega \frac{\ln \frac{1 + \tau_i F_n(T_{i-1}; T_{i-1}, T_i)}{K[1 + \tau_i F_r(0; T_{i-1}, T_i)]} - \frac{1}{2} \sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i} \sqrt{T_i - T_{i-1}}} \right) \right\}. \end{aligned} \quad (44)$$

Since the nominal forward rate $F_n(\cdot; T_{i-1}, T_i)$ follows the lognormal LIBOR model (20), we finally obtain:

$$\begin{aligned} \mathbf{ICap}(0, \mathcal{T}, \Psi, K, N, \omega) = & \omega N \psi_1 \left[P_r(0, T_1) \Phi \left(\omega \frac{\ln \frac{P_r(0, T_1)}{K P_n(0, T_1)} + \frac{1}{2} \sigma_{I,1}^2 T_1}{\sigma_{I,1} \sqrt{T_1}} \right) \right. \\ & \left. - K P_n(0, T_1) \Phi \left(\omega \frac{\ln \frac{P_r(0, T_1)}{K P_n(0, T_1)} - \frac{1}{2} \sigma_{I,1}^2 T_1}{\sigma_{I,1} \sqrt{T_1}} \right) \right] + \omega N \sum_{i=2}^M \psi_i P_n(0, T_i) \\ & \cdot \int_{-\infty}^{+\infty} \left[\frac{1 + \tau_i F_n(0; T_{i-1}, T_i) e^x}{1 + \tau_i F_r(0; T_{i-1}, T_i)} \Phi \left(\omega \frac{\ln \frac{1 + \tau_i F_n(0; T_{i-1}, T_i) e^x}{K[1 + \tau_i F_r(0; T_{i-1}, T_i)]} + \frac{1}{2} \sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i} \sqrt{T_i - T_{i-1}}} \right) \right. \\ & \left. - K \Phi \left(\omega \frac{\ln \frac{1 + \tau_i F_n(0; T_{i-1}, T_i) e^x}{K[1 + \tau_i F_r(0; T_{i-1}, T_i)]} - \frac{1}{2} \sigma_{I,i}^2 (T_i - T_{i-1})}{\sigma_{I,i} \sqrt{T_i - T_{i-1}}} \right) \right] \frac{e^{-\frac{1}{2} \left(\frac{x + \frac{1}{2} \sigma_{n,i}^2 T_{i-1}}{\sigma_{n,i} \sqrt{T_{i-1}}} \right)^2}}{\sigma_{n,i} \sqrt{2\pi T_{i-1}}} dx. \end{aligned} \quad (45)$$