PRICING and STATIC REPLICATION of FX QUANTO OPTIONS

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1 Introduction

1.1 Notation

t: the evaluation time.
τ: the running time.
S_τ: the price at time τ in domestic currency of one unit of foreign currency.
r^d_τ: the (deterministic) domestic instantaneous risk-free rate at time τ.
r^f_τ: the (deterministic) foreign instantaneous risk-free rate at time τ.
σ_τ: the exchange rate (deterministic) percentage volatility at time τ.
X: a strike price.
ω: a flag for call ($\omega = 1$) or put ($\omega = -1$).
T, T_1, T_2: future times.
Q^d: the domestic risk-neutral measure.
E^d: expectation under Q^d.
Q^N: the probability measure associated with the numeraire N.
E^N: expectation under Q^N.
$\mathcal{F}_\tau$: the σ-algebra generated by S up to time τ.
1_A: the indicator function of the set A.
$\mathrm{C}(t, T, X)$: price at time t of a (plain-vanilla) call option with maturity T and strike X.
$\mathrm{P}(t, T, X)$: price at time t of a (plain-vanilla) put option with maturity T and strike X.
$\mathrm{AoNC}(t, T, X)$: price at time t of an asset-or-nothing call with maturity T and strike X.
$\mathrm{AoNP}(t, T, X)$: price at time t of an asset-or-nothing put with maturity T and strike X.
$\mathrm{QO}(t, T, X, \omega)$: price at time t of a quanto option with maturity T and strike X.
$\mathrm{FSQO}(t, T_1, T_2, \omega)$: price at time t of a forward-start quanto option with forward-start date T_1 and maturity T_2.
$\mathrm{QCqt}(t, T_1, T_2, \omega)$: price at time t of a quanto cliquet option with forward-start date T_1 and maturity T_2.

1.2 Assumptions

The exchange rate S is assumed to evolve under the domestic risk-neutral measure Q^d according to:

$$dS_\tau = S_\tau[(r^d_\tau - r^f_\tau)\, d\tau + \sigma_\tau\, dW_\tau]$$

where W is a standard Brownian motion under Q^d. Setting $\bar{S}_\tau = S_\tau \exp\left(\int_0^\tau r_u^f \, du\right)$, the dynamics of S under the measure $Q^{\bar{S}}$ having $\bar{S}$ as numeraire is

$$dS_\tau = S_\tau[(r_\tau^d - r_\tau^f + \sigma_\tau^2)\, d\tau + \sigma_\tau \, d\bar{W}_\tau] \tag{1}$$

where $\bar{W}$ is a standard Brownian motion under $Q^{\bar{S}}$.

1.3 Pricing

The no-arbitrage price at time t of the payoff H_T at time T is

$$H_t = e^{-\int_t^T r_u^d \, du} \, E^d[H_T | \mathcal{F}_t]$$

Using $\bar{S}$ as numeraire, the time t-price becomes

$$\begin{aligned} H_t &= \bar{S}_t \, E^{\bar{S}}\left[\frac{H_T}{\bar{S}_T} | \mathcal{F}_t\right] \\ &= S_t \, e^{-\int_t^T r_u^f \, du} \, E^{\bar{S}}\left[\frac{H_T}{S_T} | \mathcal{F}_t\right] \end{aligned} \tag{2}$$

2 Quanto Options

Pricing of a Quanto Option

A quanto option pays out at maturity T the amount $[\omega(S_T - X)]^+$ in foreign currency, which is equivalent to $[\omega(S_T - X)]^+ S_T$ in domestic currency:

To price the payoff $H_T = [\omega(S_T - X)]^+ S_T$ it is convenient to use formula (2). In fact

$$\mathrm{QO}(t, T, X, \omega) = S_t \, e^{-\int_t^T r_u^f \, du} \, E^{\bar{S}}[(\omega S_T - \omega X)^+ | \mathcal{F}_t]$$

This expectation can be easily calculated under (1), since it is equivalent to an non-discounted Black-Scholes price for an underlying asset paying a continuous dividend yield $q_\tau = r_\tau^f - \sigma_\tau^2$. We thus obtain:

$$\boxed{\begin{aligned} \mathrm{QO}(t, T, X, \omega) &= \omega S_t \, e^{-\int_t^T r_u^f \, du}\left[S_t \, e^{\int_t^T (r_u^d - r_u^f + \sigma_u^2)\, du} \, \Phi(\omega d_0) - X \Phi(\omega d_1)\right] \\ d_0 &= \frac{\ln \frac{S_t}{X} + \int_t^T (r_u^d - r_u^f + \frac{3}{2}\sigma_u^2)\, du}{\sqrt{\int_t^T \sigma_u^2 \, du}} \\ d_1 &= d_0 - \sqrt{\int_t^T \sigma_u^2 \, du} \end{aligned}} \tag{3}$$

Static Replication of a Quanto Option

In the call option case, we have

$$(S_T - X)^+ S_T = \int_X^{+\infty} S_T 1_{\{S_T > K\}} \, dK = 2 \int_X^{+\infty} (S_T - K)^+ \, dK + X(S_T - X)^+ \qquad (4)$$

Therefore, a quanto call can be statically replicated by means of asset-or-nothing calls or, equivalently, plain-vanilla calls as follows:

$$\boxed{\mathrm{QO}(T, T, X, 1) = \int_X^{+\infty} \mathrm{AoNC}(T, T, K) \, dK = 2 \int_X^{+\infty} \mathrm{C}(T, T, K) \, dK + X \mathrm{C}(T, T, X)}$$

In the put option case, we have instead

$$(X - S_T)^+ S_T = \int_0^X S_T 1_{\{K > S_T\}} \, dK = X(X - S_T)^+ - 2 \int_0^X (K - S_T)^+ \, dK$$

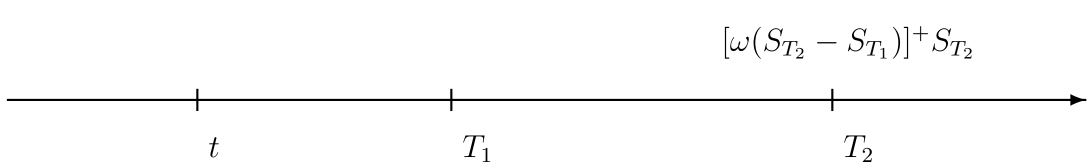
Therefore, a quanto put can be statically replicated by means of asset-or-nothing puts or, equivalently, plain-vanilla puts as follows:

$$\boxed{\mathrm{QO}(T, T, X, -1) = \int_0^X \mathrm{AoNP}(T, T, K) \, dK = X \mathrm{P}(T, T, X) - 2 \int_0^X \mathrm{P}(T, T, K) \, dK}$$

3 Forward-Start Quanto Options

Pricing of a Forward-Start Quanto Option

A forward-start quanto option pays out at maturity $T_2 > T_1$ the amount $[\omega(S_{T_2} - S_{T_1})]^+$ in foreign currency, which is equivalent to $[\omega(S_{T_2} - S_{T_1})]^+ S_{T_2}$ in domestic currency:



Since we can write

$$\mathrm{FSQO}(t, T_1, T_2, \omega) = e^{-\int_t^{T_1} r_u^d \, du} \, E^d[\mathrm{QO}(T_1, T_2, S_{T_1}, \omega) | \mathcal{F}_t]$$

using formula (3) and calculating the (risk-neutral) second moment of S_{T_1} conditional on

$\mathcal{F}_t$, we obtain

$$\boxed{\begin{aligned} \mathrm{FSQO}(t, T_1, T_2, \omega) &= \omega S_t^2 \, e^{\int_t^{T_1} (r_u^d - r_u^f + \sigma_u^2)\, du - \int_t^{T_2} r_u^f \, du} \Big[e^{\int_{T_1}^{T_2} (r_u^d - r_u^f + \sigma_u^2)\, du} \, \Phi(\omega d_0) - \Phi(\omega d_1) \Big] \\ d_0 &= \frac{\int_{T_1}^{T_2} (r_u^d - r_u^f + \frac{3}{2}\sigma_u^2)\, du}{\sqrt{\int_{T_1}^{T_2} \sigma_u^2 \, du}} \\ d_1 &= d_0 - \sqrt{\int_{T_1}^{T_2} \sigma_u^2 \, du} \end{aligned}} \tag{5}$$

Static Replication of a Forward-Start Quanto Option

The static replication of the value at time T_1 of a forward-start quanto option boils down to the static replication of $S_{T_1}^2$, both in the call and put cases. We then use (4), with $X = 0$ and $T = T_1$, thus obtaining

$$S_{T_1}^2 = \int_0^{+\infty} S_{T_1} 1_{\{S_{T_1} > K\}} \, dK = 2 \int_0^{+\infty} (S_{T_1} - K)^+ \, dK$$

Therefore, the squared exchange rate can be statically replicated by means of asset-or-nothing calls or, equivalently, plain-vanilla calls as follows:

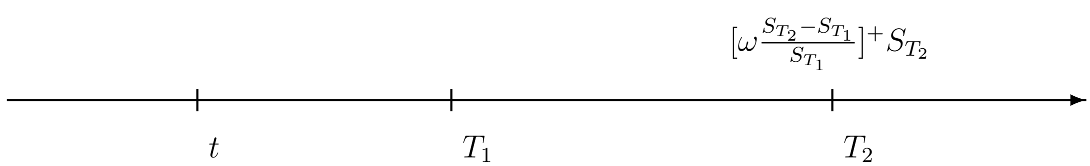
$$\boxed{S_{T_1}^2 = \int_0^{+\infty} \mathrm{AoNC}(T_1, T_1, K) \, dK = 2 \int_0^{+\infty} \mathrm{C}(T_1, T_1, K) \, dK}$$

Remark 3.1. *If the evaluation time t lies, instead, in the interval (T_1, T_2), a forward-start quanto option is equivalent to a quanto option with a given strike (the previously set S_{T_1}). We then refer to the previous section for its pricing and replication.*

4 Quanto Cliquets

Pricing of a Quanto Cliquet

A quanto cliquet option pays out at maturity $T_2 > T_1$ the amount $[\omega(S_{T_2} - S_{T_1})/S_{T_1}]^+$ in foreign currency, which is equivalent to $[\omega(S_{T_2} - S_{T_1})/S_{T_1}]^+ S_{T_2}$ in domestic currency:



Since the time T_2-payoff of a quanto cliquet is equal to that of the corresponding forward-start quanto option divided by S_{T_1}, the same applies to the corresponding values at time T_1:

$$\mathrm{QCqt}(T_1, T_1, T_2, \omega) = \frac{\mathrm{FSQO}(T_1, T_1, T_2, \omega)}{S_{T_1}}$$

By (5), the calculation of the time t-price boils down to the calculation of the (risk-neutral) expectation of S_{T_1} conditional on $\mathcal{F}_t$. We obtain

$$\boxed{\begin{aligned} \mathrm{QCqt}(t, T_1, T_2, \omega) &= \omega S_t \, e^{-\int_t^{T_2} r_u^f \, du} \Big[e^{\int_{T_1}^{T_2} (r_u^d - r_u^f + \sigma_u^2) \, du} \, \Phi(\omega d_0) - \Phi(\omega d_1) \Big] \\ d_0 &= \frac{\int_{T_1}^{T_2} (r_u^d - r_u^f + \frac{3}{2}\sigma_u^2) \, du}{\sqrt{\int_{T_1}^{T_2} \sigma_u^2 \, du}} \\ d_1 &= d_0 - \sqrt{\int_{T_1}^{T_2} \sigma_u^2 \, du} \end{aligned}} \tag{6}$$

Static Replication of a Quanto Cliquet

The quanto cliquet value at time T_1 is linear in S_{T_1}. A static replication is then achieved by buying a proper amount of foreign currency S.

Remark 4.1. *If the evaluation time t lies, instead, in the interval (T_1, T_2), a quanto cliquet is equivalent to a constant by a quanto option with a given strike, where the inverse of the constant and the strike are equal to the known value of S_{T_1}. We then refer to the related section for its pricing and replication.*