

# Smile-consistent CMS adjustments in closed form: introducing the Vanna-Volga approach

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## Abstract

In this article, we introduce the Vanna-Volga approach for an alternative valuation of CMS convexity adjustments. Our pricing procedure leads to closed-form formulas that are extremely simple to implement and that retrieve, within bid-ask spreads, market data of CMS swap spreads.

## 1 Introduction

CMS derivatives (swaps, caps and floors and spread options) are actively traded in the interest rate market and often quoted for several expiries and CMS tenors. Their popularity has urged the need for specific valuation procedures that incorporate stylized facts such as the swaption smile, and that are relatively easy to implement and manage. In particular, a systematic update of CMS swap quotes, based on any interest rate model calibrated to the swaption smile, would typically be too time consuming to be efficiently applied in practice. The purpose of this article is to propose a robust procedure that allows for a rapid and consistent calculation of CMS convexity adjustments from the quoted implied volatilities.

The pricing issues related to CMS derivatives have been studied by several authors. Hagan (2003) analyzes the pricing of CMS swaps and options relating them to the swaption market via a static replication approach and deriving closed form formulae. Berrahoui (2005) considers a different replication argument, which is based on a numerical optimization to find the optimal weights in a suitable portfolio of cash-settled swaptions. Mercurio and Pallavicini (2005, 2006) apply Hagan's replication approach by modelling implied volatilities with the SABR functional form of Hagan et al. (2002).

The SABR functional form, when calibrated to an option's smile, reveals an intrinsic parameter redundancy. Precisely, one typically finds that different values of the constant-elasticity-of-variance parameter  $\beta$  can accommodate the same smile with comparable levels of precision. To overcome this drawback, Mercurio and Pallavicini (2006) suggests to include CMS swap data in the calibration set. By doing so, one is then able to identify a unique value of  $\beta$  that fits the considered market quotes. However, the problem remains of how to derive CMS convexity adjustments with the sole knowledge of swaption smiles, and without resorting to a dynamical interest rate model.

In this article, we address this issue by presenting an alternative approach to the pricing of CMS swaps and options. Specifically, we apply the Vanna-Volga method, which is a well known empirical procedure employed by traders, especially in the FX market, to construct implied volatility surfaces and price simple exotic claims. Its consistency and robustness have been analyzed by Castagna and Mercurio (2006, 2007), who eventually hint at its possible application to the calculation of CMS convexity adjustments. Here, we follow their suggestion and show how to perform such a calculation.

The article is organized as follows. We start by briefly describing the swaption and CMS swap markets and report the smile-consistent convexity-adjustment formula, whose derivation is outlined in Mercurio and Pallavicini (2005). We then illustrate the Vanna-Volga methodology and finally show how to use it for the purpose of pricing CMS swap and options.

## 2 The swaption and CMS markets

In the current markets, besides the at-the-money swaption volatilities that are published by brokers and market makers for every expiry and tenor traded, smiles are also available for a subset of expiries and underlying tenors. Implied volatilities, expressed as a spread over the at-the-money value, are typically published for away-from-the-money strikes in terms of their distance from the at-the-money (in basis points):

$$\Delta\sigma_{a,b}^M(\Delta K) = \sigma_{a,b}^M(K^{ATM} + \Delta K) - \sigma_{a,b}^M(K^{ATM})$$

where the option has strike  $K$  and expiry  $T_a$  and is written on a swap maturing in  $T_b$ ,  $0 < T_a < T_b$ . Common distances from the at-the-money are  $\Delta K = \pm 200, \pm 100, \pm 50, \pm 25$  basis points. Clearly, suitable interpolation schemes must be devised for non-quoted strikes.

Lately, also derivatives on swap rates have become very popular. The main ones are CMS swaps, which pay the difference between a swap rate of a given maturity and a Libor rate plus a spread, typically on a quarterly basis. CMS swaps are quoted in the market as the spread over the Libor rate that makes the value of the contract nil at the inception. We then have CMS caps and floors, which are (strip of) options written on a CMS rate and analogous to the same contracts written on the Libor rates. Their premiums are quoted in basis points and they are usually traded for expiries longer than 5 years. Finally, we have CMS spread options, paying the difference between the spread of two CMS rates and a strike. Also for these contracts, premiums are expressed in basis points and common expiries are longer than 5 years, and the underlying spread is usually the 10Year-2Years.

In what follows we study the pricing of CMS swaps and CMS caps and floors. Spread options are more complex to deal with and deserve a separate treatment.

## 3 The pricing of derivatives on swap rates

We present a quick review of the basic principles to price options on swap rates. We refer to Mercurio and Pallavicini (2005, 2006) for more details.

Let us fix a maturity  $T_a$  and a set of time  $\mathcal{T}_{a,b} := \{T_{a+1}, \dots, T_b\}$  with constant year fractions  $\tau > 0$ . The forward swap rate at time  $t$  for payments in  $\mathcal{T}$  is defined as

$$S_{a,b}(t) = \frac{P(t, T_a) - P(t, T_b)}{\tau \sum_{j=a+1}^b P(t, T_j)}$$

where  $P(t, T)$  is the discount factor at time  $t$  for the maturity  $T_j$ . Denoting by  $Q^T$  the  $T$ -forward measure for a generic time  $T$  (with associated expectation  $E^T$ ), and by  $Q^{a,b}$  the forward swap measure related to  $S_{a,b}$  (with associated expectation  $E^{a,b}$ ), we have that the convexity adjustment for the swap  $S_{a,b}(T_a)$ , at time  $t = 0$ , is defined by:

$$\mathbf{CA}(S_{a,b}; \delta) = E^{T_a+\delta}[S_{a,b}(T_a)] - S_{a,b}(0), \quad (1)$$

where  $\delta \geq 0$  is the accrual period.

A CMS caplet struck at  $K$  is a call option on the swap rate  $S_{a,b}$  that pays  $[S_{a,b}(T_a) - K]^+$  at time  $T_a + \delta$ . Its price at time zero can be expressed as

$$\mathbf{CMSCpl}(S_{a,b}, K; \delta) = P(0, T_a + \delta) E^{T_a+\delta} \{(S_{a,b}(T_a) - K)^+\}. \quad (2)$$

The expectation in (1) and (2) can be calculated by moving to the forward swap measure  $Q^{a,b}$

$$E^{T_a+\delta} \{(S_{a,b}(T_a) - K)^+\} = \frac{\sum_{j=a+1}^b P(0, T_j)}{P(0, T_a + \delta)} E^{a,b} \left( \frac{(S_{a,b}(T_a) - K)^+ P(T_a, T_a + \delta)}{\sum_{j=a+1}^b P(T_a, T_j)} \right),$$

which, following Hagan's (2003) procedure, can be approximated as

$$E^{T_a+\delta} \{(S_{a,b}(T_a) - K)^+\} \approx \frac{1}{\bar{f}(S_{a,b}(0))} E^{a,b} \{\bar{f}(S_{a,b}(T_a))(S_{a,b}(T_a) - K)^+\}, \quad (3)$$

where

$$\bar{f}(S) := \frac{1}{G_{a,b}(S)(1 + \tau S)^{\frac{\delta}{\tau}}}$$

$$G_{a,b}(S) := \sum_{j=1}^{b-a} \frac{\tau}{(1 + \tau S)^j} = \begin{cases} \frac{1}{S} \left[ 1 - \frac{1}{(1 + \tau S)^{b-a}} \right] & S > 0 \\ \tau(b-a) & S = 0 \end{cases}$$

Applying standard replication arguments, one finally has:

$$E^{T_a+\delta} \{(S_{a,b}(T_a) - K)^+\} \approx \frac{1}{\bar{f}(S_{a,b}(0))} \left[ \bar{f}(K) c_{a,b}(K) + \int_K^{+\infty} [\bar{f}''(x)(x - K) + 2\bar{f}'(x)] c_{a,b}(x) dx \right], \quad (4)$$

where  $c_{a,b}(x)$  denotes the price of a payer swaption with strike  $x$ , divided by its annuity term. The CMS adjustment (1) can then be calculated by setting  $K = 0$  in (4):

$$\mathbf{CA}(S_{a,b}; \delta) \approx \frac{1}{\bar{f}(S_{a,b}(0))} \left[ \bar{f}(0) S_{a,b}(0) + \int_0^{+\infty} [\bar{f}''(x)x + 2\bar{f}'(x)] c_{a,b}(x) dx \right] - S_{a,b}(0) \quad (5)$$

As noticed by Mercurio and Pallavicini (2005, 2006), to calculate the integral in (5), one needs to introduce a model to assign a value to the swaption volatilities for strikes outside the quoted range. To this end, they employ the SABR functional form of Hagan et al. (2002), due to its popularity, tractability and ease of implementation.<sup>1</sup> In this article, we propose a different approach, which is even faster to implement and nonetheless recovers market data of CMS swap spreads with the same accuracy as the SABR form. This approach is based on the Vanna-Volga methodology described by Castagna and Mercurio (2006, 2007), which we briefly review in the following.

## 4 The Vanna-Volga Approach

The Vanna-Volga method is an empirical procedure to infer option prices, and hence implied volatilities, from three basic quotes that are available for a given maturity.

Assume that, in a given option market, three basic options are quoted for a given maturity  $T$  on an underlying asset whose initial value is  $S_0$ . We denote the corresponding strikes by  $K_i$ ,  $i = 1, 2, 3$ ,  $K_1 < K_2 < K_3$ . The market implied volatility associated to  $K_i$  is denoted by  $\sigma_i$ ,  $i = 1, 2, 3$ . The Vanna-Volga price of a European call with maturity  $T$  and strike  $K$  is defined by

$$C^{\text{VV}}(K) = C^{\text{BS}}(K) + \sum_{i=1}^3 x_i(K) [C^{\text{MKT}}(K_i) - C^{\text{BS}}(K_i)], \quad (6)$$

where  $C^{\text{BS}}(x)$  is the Black and Scholes price of the call with strike  $x$  (and maturity  $T$ ), calculated with a given volatility  $\sigma$  (typically equal to  $\sigma_2$ ),  $C^{\text{MKT}}(K_i)$  is the market price of the call with strike  $K_i$ , and the weights  $x_i$  are computed by solving the system

$$\begin{aligned} \frac{\partial C^{\text{BS}}}{\partial \sigma}(K) &= \sum_{i=1}^3 x_i(K) \frac{\partial C^{\text{BS}}}{\partial \sigma}(K_i) \\ \frac{\partial^2 C^{\text{BS}}}{\partial \sigma^2}(K) &= \sum_{i=1}^3 x_i(K) \frac{\partial^2 C^{\text{BS}}}{\partial \sigma^2}(K_i) \\ \frac{\partial^2 C^{\text{BS}}}{\partial \sigma \partial S_0}(K) &= \sum_{i=1}^3 x_i(K) \frac{\partial^2 C^{\text{BS}}}{\partial \sigma \partial S_0}(K_i) \end{aligned} \quad (7)$$

The values of  $x_i$  are found by equating the Vega, the Vanna ( $\frac{\partial \text{Vega}}{\partial S_{\text{pot}}}$ ) and the Volga ( $\frac{\partial \text{Vega}}{\partial \text{Vol}}$ ) of the call with strike  $K$  to the respective Greeks of the portfolio made of  $x_i$  units of the call with strike  $K_i$ ,  $i = 1, 2, 3$ . An explicit expression for them can be found in Castagna and Mercurio (2007).

The pricing function (6) not only defines an interpolation rule between  $K_1$  and  $K_3$  but also yields extrapolated values for option prices outside the interval  $[K_1, K_3]$ . This consideration led Castagna and Mercurio (2007) to propose the possible use of the Vanna-Volga

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<sup>1</sup>Mercurio and Pallavicini (2005) also consider the example of a mixture of shifted-lognormal densities.

approach to model swaption volatilities outside the strike range quoted by the market, so as to produce an alternative valuation of the integral in (5).

Now, we show how to apply the Vanna-Volga method for the calculation of CMS convexity adjustments.

## 5 The Vanna-Volga convexity adjustment formula

Assume that, for the swap rate  $S_{a,b}$ , volatility quotes are available for the at-the-money strike  $K_2 = S_{a,b}(0)$  and for two away-from-the-money strikes  $K_1$  and  $K_3$ , respectively below and above  $S_{a,b}(0)$ . Setting  $\sigma$  equal to the at-the-money volatility  $\sigma_2 = \sigma_{\text{ATM}}$  (market quote for  $K_2$ ), formula (6) can be re-written in the swaption case as follows:

$$c_{a,b}^{\text{VV}}(K) = c_{a,b}^{\text{BS}}(K) + \sum_{i=1}^3 x_i(K) [c_{a,b}^{\text{MKT}}(K_i) - c_{a,b}^{\text{BS}}(K_i)], \quad (8)$$

where

$$\begin{aligned} c_{a,b}^{\text{BS}}(K) &:= \text{Bl}(K, S_{a,b}(0), \sigma_{\text{ATM}} \sqrt{T_a}), \\ c_{a,b}^{\text{MKT}}(K_i) &:= \text{Bl}(K_i, S_{a,b}(0), \sigma_i \sqrt{T_a}), \\ \text{Bl}(K, S, v) &:= S \Phi\left(\frac{\ln(S/K) + v^2/2}{v}\right) - K \Phi\left(\frac{\ln(S/K) - v^2/2}{v}\right), \end{aligned}$$

with  $\Phi$  denoting the standard normal cumulative distribution function, and the weights  $x_i(K)$  are explicitly given, see Castagna and Mercurio (2007), by

$$\begin{aligned} x_1(K) &= \frac{\mathcal{V}(K)}{\mathcal{V}(K_1)} \frac{\ln \frac{K_2}{K} \ln \frac{K_3}{K}}{\ln \frac{K_2}{K_1} \ln \frac{K_3}{K_1}}, \\ x_2(K) &= \frac{\mathcal{V}(K)}{\mathcal{V}(K_2)} \frac{\ln \frac{K}{K_1} \ln \frac{K_3}{K}}{\ln \frac{K_2}{K_1} \ln \frac{K_3}{K_2}}, \\ x_3(K) &= \frac{\mathcal{V}(K)}{\mathcal{V}(K_3)} \frac{\ln \frac{K}{K_1} \ln \frac{K}{K_2}}{\ln \frac{K_3}{K_1} \ln \frac{K_3}{K_2}}, \end{aligned}$$

with  $\mathcal{V}(K)$  the Black and Scholes Vega (divided by the annuity) of the payer swaption with strike  $K$ :

$$\begin{aligned} \mathcal{V}(K) &:= S_{a,b}(0) \sqrt{T_a} \varphi\left(\frac{\ln \frac{S_{a,b}(0)}{K} + \frac{1}{2} \sigma_{\text{ATM}}^2 T_a}{\sigma_{\text{ATM}} \sqrt{T_a}}\right), \\ \varphi(x) &:= \Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}. \end{aligned}$$

Therefore, the Vanna-Volga convexity adjustment (1) for the swap rate  $S_{a,b}$  can be calculated as

$$\mathbf{CA}^{\text{VV}}(S_{a,b}; \delta) \approx \frac{1}{\bar{f}(S_{a,b}(0))} \left[ \bar{f}(0)S_{a,b}(0) + \int_0^{+\infty} [\bar{f}''(x)x + 2\bar{f}'(x)]c_{a,b}^{\text{VV}}(x) dx \right] - S_{a,b}(0). \quad (9)$$

More generally, the Vanna-Volga price of a CMS caplet is given by

$$\begin{aligned} \mathbf{CMSCpl}^{\text{VV}}(S_{a,b}, K; \delta) &= P(0, T_a + \delta) E^{T_a + \delta} \{(S_{a,b}(T_a) - K)^+\} \\ &\approx \frac{P(0, T_a + \delta)}{\bar{f}(S_{a,b}(0))} \left[ \bar{f}(K)c_{a,b}^{\text{VV}}(K) + \int_K^{+\infty} [\bar{f}''(x)(x - K) + 2\bar{f}'(x)]c_{a,b}^{\text{VV}}(x) dx \right]. \end{aligned} \quad (10)$$

Formulas (9) and (10) provide alternative valuations of convexity adjustments and CMS options. Their computational complexity is comparable to those obtained with the SABR functional form. However, contrary to SABR case, for which equivalent smile-fitted parameters can imply quite different adjustments, the Vanna-Volga method is more consistent. Once assigned the basic strikes, (9) and (10) can be calculated with no ambiguity.

The goodness of the VV approach for the valuation of CMS adjustments must also be tested on market data. This is achieved in the following.

## 6 An example with market data

For our testing purposes, we first retrieve the swaption volatilities for different strikes from the published ones (we use Bloomberg pages). For each expiry, we choose three strikes: the ATM (equal to the forward swap rate) and two wings (respectively lower and higher than the ATM) such that the  $\Delta$  of the swaption (again, without any inclusion of the annuity) is as close as possible to 25% (from below), in absolute terms, both for the payer (higher strike) and the receiver (lower strike).

After identifying the basic strikes and the related implied volatilities, we follow the approach described above, computing the integral in (9) by Vanna-Volga interpolation of the market volatility surface as of January 26th, 2007. We have applied this methodology to evaluate CMS swap spreads on the same date. In Table 1 we present market data for 5 different CMS indexes (2y, 5y, 10y, 20y, 30y) and 5 different swap maturities (5y, 10y, 15y, 20y, 30y).

Results show that the Vanna-Volga approach performs quite well. Differences with respect to market mid values are typically much lower than one basis point and always inside the bid-ask spreads (the largest differences we observe correspond to the largest bid-ask spreads). Such differences are displayed in Figure 1.

## 7 Simplifying the calculations

As already noticed, the Vanna-Volga approach involves the numerical computation of an integral, which as such is rather time consuming. Contrary to the SABR case, however,

<b>CMS-2y</b> Maturity	<b>Bid</b>	<b>Ask</b>	<b>Mid</b>	<b>VV</b>	<b>CMS-5y</b> Maturity	<b>Bid</b>	<b>Ask</b>	<b>Mid</b>	<b>VV</b>
<b>5y</b>	16.4	16.9	16.7	16.5	<b>5y</b>	21.2	22.2	21.7	21.6
<b>10y</b>	18.4	19.6	19.0	18.9	<b>10y</b>	25.8	27.0	26.4	26.2
<b>15y</b>	18.8	20.1	19.5	19.4	<b>15y</b>	25.5	27.4	26.5	26.3
<b>20y</b>	18.5	20.0	19.3	19.4	<b>20y</b>	24.5	26.8	25.7	25.7
<b>30y</b>	18.5	20.5	19.5	19.6	<b>30y</b>	24.7	27.9	26.3	26.0

<b>CMS-10y</b> Maturity	<b>Bid</b>	<b>Ask</b>	<b>Mid</b>	<b>VV</b>	<b>CMS-20y</b> Maturity	<b>Bid</b>	<b>Ask</b>	<b>Mid</b>	<b>VV</b>
<b>5y</b>	32.0	34.0	33.0	32.9	<b>5y</b>	41.7	44.5	43.1	42.7
<b>10y</b>	36.5	38.5	37.5	37.1	<b>10y</b>	43.5	46.5	45.0	44.2
<b>15y</b>	35.4	37.4	36.4	35.6	<b>15y</b>	39.8	44.1	42.0	41.1
<b>20y</b>	33.5	35.5	34.5	34.0	<b>20y</b>	37.4	42.5	40.0	39.5
<b>30y</b>	33.0	37.1	35.1	34.0	<b>30y</b>	39.8	47.1	43.5	42.9

<b>CMS-30y</b> Maturity	<b>Bid</b>	<b>Ask</b>	<b>Mid</b>	<b>VV</b>
<b>5y</b>	39.3	43.1	41.2	40.6
<b>10y</b>	41.4	45.4	43.4	42.0
<b>15y</b>	37.3	43.2	40.3	39.2
<b>20y</b>	34.7	41.5	38.1	38.1
<b>30y</b>	38.4	48.2	43.4	43.1

Table 1: Market values for CMS swap spreads compared to the corresponding Vanna-Volga ones.

we can resort to a closed-form approximation that considerably speeds up the pricing procedure.

In fact, the calculation of the adjustment (9) and caplet price (10) can be simplified by using a robustness property that holds for the implied volatilities, and hence option prices, constructed with the Vanna-Volga method. Precisely, we refer to the second consistency result stated and proved by Castagna and Mercurio (2006, 2007), which reads as follows: a European-style payoff  $h(S_{a,b}(T_a))$ , at time  $T_a + \delta$ , can be equivalently valued either by using a static-replication argument combined with the explicit formula for the Vanna-Volga option prices (8) or by applying the Vanna-Volga construction procedure directly to the payoff to price.

Therefore, in our CMS case, instead of using the Vanna-Volga price for the options  $c_{a,b}(x)$  in (4), leading to formulas (9) and (10), one can calculate the expectation in the RHS of (3) by applying the Vanna-Volga method directly to the European-style payoff  $H := \bar{f}(S_{a,b}(T_a))(S_{a,b}(T_a) - K)^+$ . We obtain:

$$E^{a,b}\{\bar{f}(S_{a,b}(T_a))(S_{a,b}(T_a) - K)^+\} = \Pi^{H,vv} := \Pi^{H,BS} + \sum_{i=1}^3 x_i^H [c_{a,b}^{\text{MKT}}(K_i) - c_{a,b}^{\text{BS}}(K_i)], \quad (11)$$

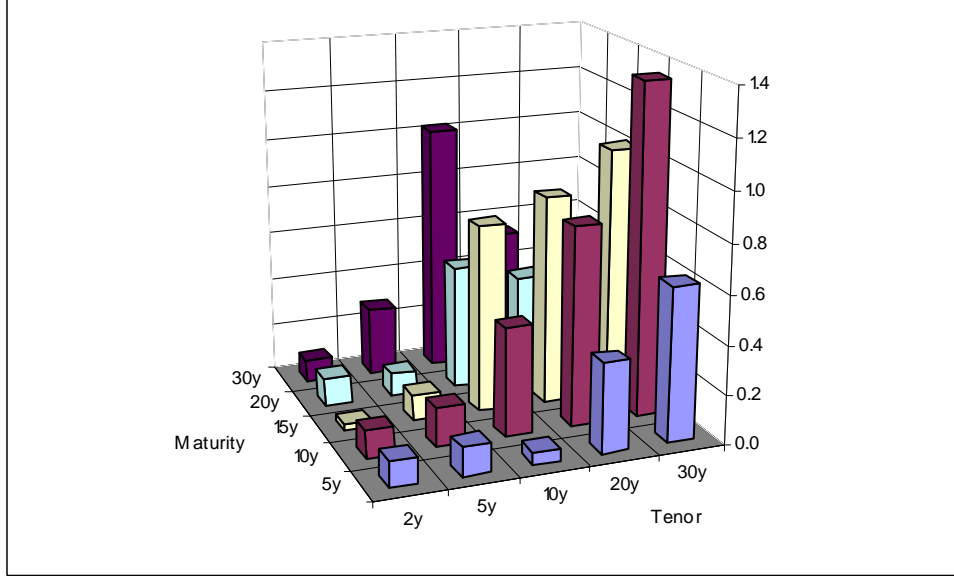


Figure 1: Differences (in absolute value) between market CMS swap spreads and Vanna-Volga ones.

where we again set  $\sigma = \sigma_2 = \sigma_{\text{ATM}}$ ,  $\Pi^{H,\text{BS}}$  is the time-0 price of the payoff  $H$  at time  $T_a + \delta$  under Black-Scholes' dynamics, *i.e.*

$$\begin{aligned} \Pi^{H,\text{BS}} &= \int_0^{+\infty} \bar{f}(x)(x-K)^+ \frac{d^2 c_{a,b}^{\text{BS}}}{dx^2}(x) dx \\ &= \bar{f}(K)c_{a,b}^{\text{BS}}(K) + \int_K^{+\infty} [\bar{f}''(x)(x-K) + 2\bar{f}'(x)] c_{a,b}^{\text{BS}}(x) dx, \end{aligned} \quad (12)$$

and the weights  $x_i^H$  are the unique solution to the system

$$\begin{aligned} \frac{\partial \Pi^{H,\text{BS}}}{\partial \sigma} &= \sum_{i=1}^3 x_i^H \frac{\partial c_{a,b}^{\text{BS}}}{\partial \sigma}(K_i) \\ \frac{\partial^2 \Pi^{H,\text{BS}}}{\partial \sigma^2} &= \sum_{i=1}^3 x_i^H \frac{\partial^2 c_{a,b}^{\text{BS}}}{\partial \sigma^2}(K_i) \\ \frac{\partial^2 \Pi^{H,\text{BS}}}{\partial \sigma \partial S_{a,b}(0)} &= \sum_{i=1}^3 x_i^H \frac{\partial^2 c_{a,b}^{\text{BS}}}{\partial \sigma \partial S_{a,b}(0)}(K_i) \end{aligned} \quad (13)$$

Analogously to the Vanna-Volga swaption price (8), the Vanna-Volga price of the payoff  $H$  is obtained by adding to the (Black-Scholes) flat-smile value of the derivative  $H$  the weighted sum of the differences between the market value of each traded option  $c_{a,b}^{\text{MKT}}(K_i)$  and its Black-Scholes value  $c_{a,b}^{\text{BS}}(K_i)$ . The weights  $x_i^H$  are now computed by equating the



Vega, the Vanna and the Volga of the derivative's price  $\Pi^{H,BS}$  to the corresponding Greeks of the portfolio built with the three basic swaptions.<sup>2</sup>

The calculation of  $\Pi^{H,vv}$  in (11) is based on the numerical calculation of either integral in (12) and the numerical solution of the linear system (13). Therefore, the robustness result we just applied does not seem to be that helpful, since the computational effort required for (11) is comparable with that for (10). However, a clear improvement, as far as computation time is concerned, can be achieved by expanding the function  $\bar{f}$  around its known value  $\bar{f}(S_{a,b}(0))$  up to a given order  $n$ :

$$\bar{f}(x) \approx \tilde{f}_n(x) := \sum_{i=0}^n \frac{\bar{f}^{(i)}(S_{a,b}(0))}{i!} [x - S_{a,b}(0)]^i. \quad (14)$$

By doing so, we can calculate  $\Pi^{H,BS}$  (approximately) as follows

$$\Pi^{H,BS} \approx \int_K^{+\infty} \tilde{f}_n(x) (x - K) g_{a,b}^{BS}(x) dx, \quad (15)$$

with

$$g_{a,b}^{BS}(x) := \frac{d^2 c_{a,b}^{BS}}{dx^2}(x) = \frac{1}{x \sigma_{ATM} \sqrt{T_a}} \varphi\left(\frac{\ln \frac{S_{a,b}(0)}{x} - \frac{1}{2} \sigma_{ATM}^2 T_a}{\sigma_{ATM} \sqrt{T_a}}\right),$$

which can be obtained in closed form, being equal to the integral of an algebraic function times a lognormal density.

For practical purposes, it is typically enough to choose  $n = 3$ . See also Figure 3 below for a graphical comparison of approximations at different orders. Hereafter, therefore, we expand  $\bar{f}$  up to the third order, replacing  $\bar{f}$  with  $\tilde{f}_3$ . To this end, the first three derivatives of  $\bar{f}$  at point  $S_{a,b}(0)$  are explicitly given by

$$\begin{aligned} \bar{f}'(S_{a,b}(0)) &= \frac{\theta}{S_{a,b}(0)} \bar{f}(S_{a,b}(0)) \\ \bar{f}''(S_{a,b}(0)) &= \frac{\theta^2 - \gamma}{S_{a,b}^2(0)} \bar{f}(S_{a,b}(0)) \\ \bar{f}'''(S_{a,b}(0)) &= \frac{\theta^3 - 3\theta\gamma + 2\eta}{S_{a,b}^3(0)} \bar{f}(S_{a,b}(0)) \end{aligned} \quad (16)$$

where, setting  $l := b - a$  and  $T(S_{a,b}(0)) := (1 + \tau S_{a,b}(0))^l - 1$ ,

$$\begin{aligned} \theta &:= 1 - \frac{\tau S_{a,b}(0)}{1 + \tau S_{a,b}(0)} \left( \frac{\delta}{\tau} + \frac{l}{T(S_{a,b}(0))} \right) \\ \gamma &:= 1 - \frac{(\tau S_{a,b}(0))^2}{(1 + \tau S_{a,b}(0))^2} \left( \frac{\delta}{\tau} + \frac{l + l^2}{T(S_{a,b}(0))} + \frac{l^2}{T^2(S_{a,b}(0))} \right) \\ \eta &:= 1 - \frac{(\tau S_{a,b}(0))^3}{(1 + \tau S_{a,b}(0))^3} \left( \frac{\delta}{\tau} + \frac{l + \frac{3}{2}l^2 + \frac{1}{2}l^3}{T(S_{a,b}(0))} + \frac{\frac{3}{2}l^2 + \frac{3}{2}l^3}{T^2(S_{a,b}(0))} + \frac{l^3}{T^3(S_{a,b}(0))} \right) \end{aligned} \quad (17)$$

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<sup>2</sup>The terms prices and Greeks are here used with a slight abuse of terminology. In fact, actual prices and Greeks are obtained by multiplying the former quantities by the “annuity term”  $1/\bar{f}(S_{a,b}(0))$ .

When  $K = 0$ , the calculation of (15) boils down to the calculation of the first  $n+1$  moments of a lognormal density. We get:

$$\begin{aligned} \Pi^{H,BS}|_{K=0} \approx & S_{a,b}(0) \left( 1 - AS_{a,b}(0) + \frac{1}{2}BS_{a,b}^2(0) - \frac{1}{6}CS_{a,b}^3(0) \right) + 2AI_1 \\ & + B(3I_2 - 2S_{a,b}(0)I_1) + C(2I_3 - 3S_{a,b}(0)I_2 + S_{a,b}^2(0)I_1) \end{aligned} \quad (18)$$

where

$$\begin{aligned} A &= \frac{\theta}{S_{a,b}(0)} \\ B &= \frac{\theta^2 - \gamma}{S_{a,b}^2(0)} \\ C &= \frac{\theta^3 - 3\gamma\theta + 2\eta}{S_{a,b}^3(0)} \\ I_1 &= \frac{1}{2}S_{a,b}^2(0)e^{\sigma^2 T_a} \\ I_2 &= \frac{1}{6}S_{a,b}^3(0)e^{3\sigma^2 T_a} \\ I_3 &= \frac{1}{12}S_{a,b}^4(0)e^{6\sigma^2 T_a} \end{aligned} \quad (19)$$

The simplified expression of the Vanna-Volga convexity adjustment for the swap rate  $S_{a,b}$  is then obtained by combining (11) with (18), setting  $K = 0$ :

$$\begin{aligned} \mathbf{CA}^{VV}(S_{a,b}; \delta) \approx & \frac{1}{\bar{f}(S_{a,b}(0))} \left[ S_{a,b}(0) \left( 1 - AS_{a,b}(0) + \frac{1}{2}BS_{a,b}^2(0) - \frac{1}{6}CS_{a,b}^3(0) \right) + 2AI_1 \right. \\ & + B(3I_2 - 2S_{a,b}(0)I_1) + C(2I_3 - 3S_{a,b}(0)I_2 + S_{a,b}^2(0)I_1) \\ & \left. + \sum_{i=1}^3 x_i^H|_{K=0} (c_{a,b}^{\text{MKT}}(K_i) - c_{a,b}^{\text{BS}}(K_i)) \right] - S_{a,b}(0), \end{aligned} \quad (20)$$

where  $x_i^H|_{K=0}$  are the weights when  $K = 0$ .

As to the Vanna-Volga pricing of a CMS caplet, tedious but straightforward algebra leads to:

$$\begin{aligned} \Pi^{H,BS} = & c_{a,b}(K) \left[ 1 + A(K - S_{a,b}(0)) + \frac{B}{2}(K - S_{a,b}(0))^2 + \frac{C}{6}(K - S_{a,b}(0))^3 \right] + 2AJ_1 \\ & + B[3J_2 - (K + 2S_{a,b}(0))J_1] + C[2J_3 - (K + 3S_{a,b}(0))J_2 + (S_{a,b}^2(0) + S_{a,b}(0)K)J_1] \end{aligned} \quad (21)$$

where

$$\begin{aligned} J_1 &:= \frac{1}{2} \left[ S_{a,b}^2(0)e^{\sigma^2 T_a} \Phi(d_{\frac{3}{2}}) - 2KS_{a,b}(0)\Phi(d_{\frac{1}{2}}) + K^2\Phi(d_{-\frac{1}{2}}) \right] \\ J_2 &:= \frac{1}{6} \left[ S_{a,b}^3(0)e^{3\sigma^2 T_a} \Phi(d_{\frac{5}{2}}) - 3K^2S_{a,b}(0)\Phi(d_{\frac{1}{2}}) + 2K^3\Phi(d_{-\frac{1}{2}}) \right] \\ J_3 &:= \frac{1}{12} \left[ S_{a,b}^4(0)e^{6\sigma^2 T_a} \Phi(d_{\frac{7}{2}}) - 4K^3S_{a,b}(0)\Phi(d_{\frac{1}{2}}) + 3K^4\Phi(d_{-\frac{1}{2}}) \right] \end{aligned}$$

and

$$d_q := \frac{\ln \frac{S_{a,b}(0)}{K} + q\sigma_{\text{ATM}}^2 T_a}{\sigma_{\text{ATM}} \sqrt{T_a}}.$$

Therefore, we finally obtain

$$\begin{aligned} \text{CMSCplt}^{\text{vv}}(S_{a,b}, K; \delta) &\approx \frac{P(0, T_a + \delta)}{\bar{f}(S_{a,b}(0))} \left[ c_{a,b}(K) \left( 1 + A(K - S_{a,b}(0)) + \frac{B}{2}(K - S_{a,b}(0))^2 \right. \right. \\ &\quad \left. \left. + \frac{C}{6}(K - S_{a,b}(0))^3 \right) + 2AJ_1 + B(3J_2 - (K + 2S_{a,b}(0))J_1) \right. \\ &\quad \left. + C(2J_3 - (K + 3S_{a,b}(0))J_2 + (S_{a,b}^2(0) + S_{a,b}(0)K)J_1) \right. \\ &\quad \left. + \sum_{i=1}^3 x_i^H (c_{a,b}^{\text{MKT}}(K_i) - c_{a,b}^{\text{BS}}(K_i)) \right], \end{aligned} \tag{22}$$

where  $x_i^H$  are the weights for the generic strike  $K > 0$ .

## 8 Numerical examples on CMS swaps

In this section we will compare the performance of the third-order approximation (20) with the full (numerical) calculation of the integral in (9). If the approximation works nicely, then the calculation time for the Vanna-Volga procedure can be reduced drastically. Besides, we will compare the Vanna-Volga prices (third-order approximation) with those produced by the SABR model as in Mercurio and Pallavicini (2005, 2006), since this latter approach is often used by practitioners in the swaption market.

In Table 2, we show the mid market prices and the Vanna-Volga prices obtained via third-order approximation. We find that the approximated Vanna-Volga prices lie all within the market bid-ask spreads as reported in Table 1. Moreover, the comparison with the values coming from formula (9), see again Table 1, shows that the third-order approximation works fairly well.<sup>3</sup> The differences between exact and approximated prices are reported in the third column of Table 2 and graphically represented in Figure 2. The discrepancies in values are indeed negligible, with the largest differences observed for long tenors and maturities, for which the third-order expansion tends to deviate from the true value of  $\bar{f}$  (see Figure 3 where the case of a 10y CSM rate is considered).

In Table 2, we also include the CMS swap spreads coming from the SABR functional form calibrated to market volatilities. We choose different  $\beta$  parameters for different CMS tenors:  $\beta = 0.5$  for CMS-2y,  $\beta = 0.6$  for CMS-5y, CMS-20y and CMS-30y, and  $\beta = 0.7$  for CMS-10y. In fact, as pointed out by Mercurio and Pallavicini (2006), different  $\beta$ 's can accommodate the swaption smile with similar accuracy, but the implied CMS adjustments

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<sup>3</sup>The third order seems to yield a good trade-off between performance and precision. Typically, the second order approximation is not enough satisfactory, and adding the fourth order gives only a slight improvement, which does not justify the increase in complexity.

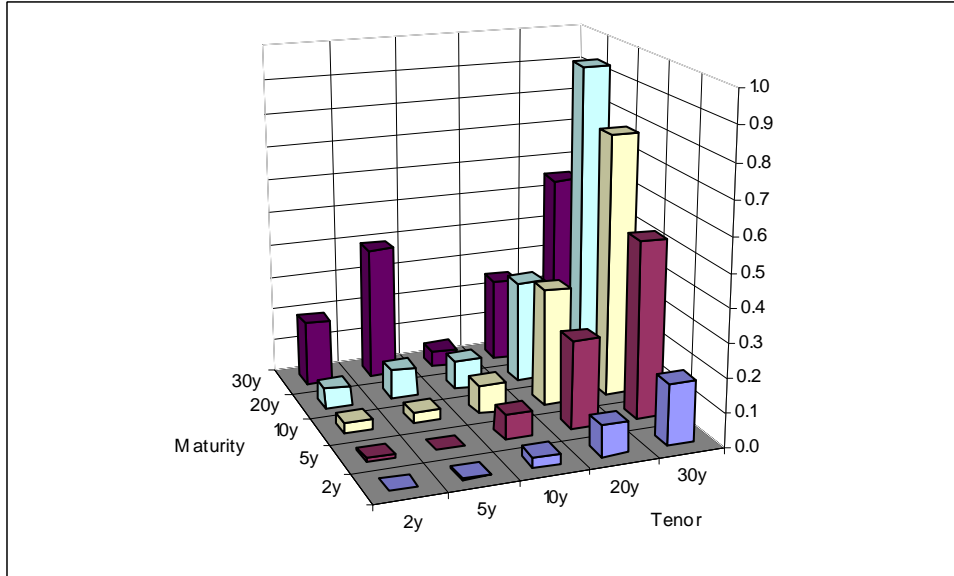


Figure 2: Absolute differences between CMS spreads computed with the general and the simplified versions of the Vanna-Volga approach.

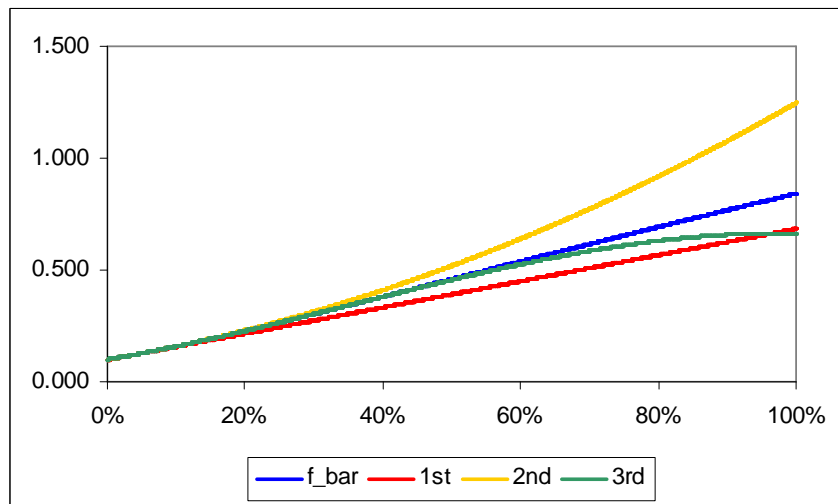


Figure 3: Comparison between  $\bar{f}$  with its approximations up to the third order.

<b>CMS-2y</b> Maturity	<b>Mkt</b>	<b>VV<sup>3rd</sup></b>	<b><math>\Delta_{VV}</math></b>	<b>SABR</b>	<b>CMS-5y</b> Maturity	<b>Mkt</b>	<b>VV<sup>3rd</sup></b>	<b><math>\Delta_{VV}</math></b>	<b>SABR</b>
<b>5y</b>	16.7	16.5	0.0	16.5	<b>5y</b>	21.7	21.6	0.0	21.6
<b>10y</b>	19.0	18.9	0.0	18.9	<b>10y</b>	26.4	26.2	0.0	26.2
<b>15y</b>	19.5	19.4	0.0	19.4	<b>15y</b>	26.5	26.3	0.0	26.3
<b>20y</b>	19.3	19.3	0.1	19.4	<b>20y</b>	25.7	25.7	0.0	25.7
<b>30y</b>	19.5	19.4	0.2	19.4	<b>30y</b>	26.3	25.6	0.4	26.0

<b>CMS-10y</b> Maturity	<b>Mkt</b>	<b>VV<sup>3rd</sup></b>	<b><math>\Delta_{VV}</math></b>	<b>SABR</b>	<b>CMS-20y</b> Maturity	<b>Mkt</b>	<b>VV<sup>3rd</sup></b>	<b><math>\Delta_{VV}</math></b>	<b>SABR</b>
<b>5y</b>	33.0	32.9	0.0	33.0	<b>5y</b>	43.1	42.8	-0.1	42.8
<b>10y</b>	37.5	37.1	0.0	37.4	<b>10y</b>	45.0	44.4	-0.2	44.8
<b>15y</b>	36.4	35.7	-0.1	35.0	<b>15y</b>	42.0	41.4	-0.3	42.3
<b>20y</b>	34.5	34.1	-0.1	35.0	<b>20y</b>	40.0	39.8	-0.3	41.0
<b>30y</b>	35.1	34.0	0.0	35.3	<b>30y</b>	43.5	42.7	0.2	44.6

<b>CMS-30y</b> Maturity	<b>Mkt</b>	<b>VV<sup>3rd</sup></b>	<b><math>\Delta_{VV}</math></b>	<b>SABR</b>
<b>5y</b>	41.2	40.8	-0.2	40.7
<b>10y</b>	43.4	42.6	-0.6	42.8
<b>15y</b>	40.3	40.0	-0.8	40.7
<b>20y</b>	38.1	39.1	-1.0	39.7
<b>30y</b>	43.3	43.6	-0.5	44.2

Table 2: Comparison between market CMS swap spreads and those computed with the approximated Vanna-Volga method ( $VV^{3rd}$ ) and the SABR model.  $\Delta_{VV}$  denotes the difference between exact and approximated Vanna-Volga spreads (in basis points).

can be quite different. The  $\beta$  values we use in our calibration are then chosen so as to reproduce also CMS data in a satisfactory way.

In Figure 4 we plot the (absolute) differences between CMS swap spreads computed with the SABR and the Vanna-Volga approaches. The discrepancies between the two models are very small for short-dated swaps, while they increase for higher swap maturities and tenors. Again, differences are negligible when compared to bid-ask spreads.

## 9 Numerical examples on CMS caps

As a last numerical example, we consider the pricing of CMS caps with different methods. Precisely, we compare the values obtained through the Vanna-Volga formula (10) with those coming from (4) combined with SABR implied volatilities. We also show the results obtained with the approximated Vanna-Volga prices (22) and those implied by the approach presented in Hagan (2003).

In Table 3 we report the prices of a 10 year cap on the CMS-10y index for different

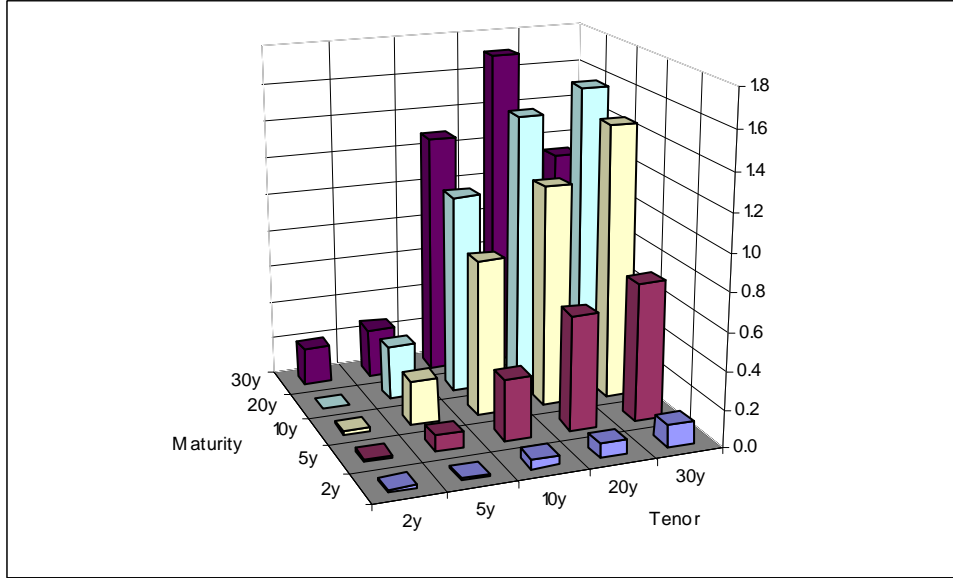


Figure 4: Comparison between SABR and Vanna-Volga CMS swap spreads after calibration to swaption data.

Strike	VV	VV <sup>3rd</sup>	SABR	Hagan
2.5%	2095.5	2107.6	2112.4	2191.3
4.5%	393.0	396.7	393.8	385.6
10.0%	11.5	11.2	11.0	8.3

Table 3: Prices (in basis points) of 10 years CMS caps on the CMS-10y index.

strikes: one deep in-the-money (2%), one nearly at-the-money (4.5%) and one deep out-of-the-money (10%). Prices are calculated with the model parameters obtained through the calibration outlined in the previous section.

The Vanna-Volga approach (both in the general and in the approximated versions) and the SABR-based one lead to quite similar results, particularly for the ATM case.

On the other side, Hagan’s method yields rather different values. This is not surprising since these latter prices are not obtained using the whole volatility smile, but a volatility quote only, namely the one corresponding to the particular strike considered.

## 10 Conclusions

The Vanna-Volga approach provides alternative valuations of CMS convexity adjustments and options, with similar computational complexity of the popular method based on SABR implied volatilities.

However, the Vanna-Volga approach has clear advantages. First, in a swaption smile construction, it only requires three quoted strikes to build a consistent implied volatility curve, without involving any calibration procedure and without assigning a priori any model parameter. Second, in the valuation of CMS adjustments, it allows for an analytical approximation that fasten considerably the pricing procedure. The approximation is extremely accurate and fast, involving no numerical integration.

The Vanna-Volga approach can also be used for the pricing of European-style derivatives on a single CMS rate. Derivatives with payoffs depending on two or more CMS rates, like spread options for instance, are more complex to deal with, since the extension of the Vanna-Volga methodology to the multi-asset case is not straightforward.

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